

# LINEAR POTENTIALS IN GALAXIES AND CLUSTERS OF GALAXIES

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## Abstract

In a previous paper we presented a typical set of galactic rotation curves associated with the linear gravitational potential of the conformal invariant fourth order theory of gravity which has recently been advanced by Mannheim and Kazanas as a candidate alternative to the standard second order Newton-Einstein theory. Reasonable agreement with data was obtained for four representative galaxies without the need for any non-luminous or dark matter. In this paper we present the associated formalism and compare and contrast the linear potential explanation of the general systematics of galactic rotation curves and the associated Tully-Fisher relation with that of the standard dark matter theory. Additionally, we show that the conformal gravity picture appears to have survived the recent round of microlensing observations unscathed. Finally, we make a first application of the conformal theory to the larger distance scale associated with a cluster of galaxies, with the theory being found to give a reasonable value for the mean velocity of the virialized core of the typical Coma cluster, again without the need to invoke dark matter.

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## (1) Introduction

At the present time there is little doubt in the general community as to the correctness of the standard second order Newton-Einstein theory of gravity. However given the fact that the application of this theory to currently available astrophysical and cosmological data obliges the Universe to be composed of overwhelming amounts of non-luminous or dark matter, it is a well established scientific tradition to pause and question so startling an implication, and to at least consider the possibility that this need for dark matter might instead actually be signaling a possible breakdown of the standard theory. Since this apparent need for dark matter is manifest on essentially every single distance scale from galactic all the way up to cosmological, while no such need is generally manifest on the much shorter distance scales where the standard gravitational theory was originally established in the first place, it is thus natural to consider the possibility that new physics (one might even refer to it as dark physics) may be opening up on these bigger distance scales. Indeed, rather than interpreting essentially every single current large distance scale gravitational observation as yet further evidence for the existence of dark matter (the common practice in both the learned and the popular literature), these selfsame data can just as equally be regarded as signaling the repeated failure of the standard theory; and definitively so if the bulk of the matter which actually exists in the Universe is in fact luminously observable. Thus the psychologically unwelcome empirical possibility suggested by the data (and now all the more so given the apparent failure so far of the current round of direct microlensing and optical dark and faint matter searches to validate the standard galactic spherical dark halo scenario) is that Newton's Law of Gravity may not be the correct weak gravitational theory on large distance scales, and that, accordingly, second order Einstein gravity may not then be the correct covariant one.

Now of course both the Newton and Einstein theories enjoy many successes (enough to convince most people that they are no longer even challengeable at all), and thus any alternate theory of gravity must be able to recover all their established features. To achieve this, one way to proceed is to begin with galactic rotation curve data (perhaps the most clear cut and well explored situation where the Newton-Einstein theory demands dark matter) and try to extract out a new weak gravity limit which encompasses Newton in an appropriate limit (see e.g. Milgrom 1983a, b, c and Sanders 1990) with a view to then subsequently working upwards to a covariant generalization (a program which is still in progress - see Bekenstein 1987 and Sanders 1990 for recent reviews). However, in order to ensure encompassing the Einstein successes from the outset, there is also much merit in beginning covariantly and then working downwards to a weak gravity limit (an approach which then comes with the additional challenge of not knowing what weak gravity limit may eventually ensue until after solutions to any candidate alternate covariant theory have actually been found), with this latter approach having actually been advanced and explored in the literature by Mannheim and Kazanas in a recent series of papers (Mannheim and Kazanas 1989, 1991, 1994, Mannheim 1990, 1992, 1993a, 1993b, 1994, Kazanas 1991, Kazanas and Mannheim 1991a, 1991b). Recognizing that there is currently no known theoretical reason which would select out the standard second order Einstein theory from amongst the infinite class of (all order) covariant, pure metric based theories of gravity that one could in principle at least consider, Mannheim and Kazanas reopened the question of what the correct covariant theory might then be and developed an approach which fixes the gravitational action by imposing an additional fundamental principle above and beyond covariance, namely that of local scale or conformal invariance, i.e. invariance under any and all local conformal stretchings  $g_{\mu\nu}(x) \rightarrow \Omega^2(x)g_{\mu\nu}(x)$  of the geometry, this being the invariance which is now believed to be possessed by the other three fundamental interactions, the strong, the electromagnetic and the weak. This invariance forces gravity to then be described uniquely by the fourth order action

$$I_W = -\alpha \int d^4x (-g)^{1/2} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa} = -2\alpha \int d^4x (-g)^{1/2} (R_{\lambda\mu} R^{\lambda\mu} - (R^\alpha{}_\alpha)^2/3) \quad (1)$$

where  $C_{\lambda\mu\nu\kappa}$  is the conformal Weyl tensor and  $\alpha$  is a purely dimensionless coefficient. In their original paper Mannheim and Kazanas (1989) obtained the complete and exact, non-perturbative exterior vacuum solution associated with a static, spherically symmetric gravitational source such as a star in this theory, viz.

$$-g_{00} = 1/g_{rr} = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \gamma r - kr^2 \quad (2)$$

(for line element  $ds^2 = -g_{00}c^2dt^2 - g_{rr}dr^2 - r^2d\Omega$ ) where  $\beta, \gamma$ , and  $k$  are three appropriate dimensionful integration constants. As can be seen, for small enough values of the linear and quadratic terms (i.e. on small enough distance scales) the solution reduces to the familiar Schwarzschild solution of Einstein gravity, with the conformal theory then enjoying the same static successes as the Einstein theory on those distance scales. On larger distance scales, however, the theory begins to differ from the Einstein theory through the linear potential term, and (with the quadratic term only possibly being important cosmologically, and with both the  $\beta\gamma$  product terms being found to be numerically negligible in the fits of Mannheim 1993b) then yields a non-relativistic gravitational potential

$$V(r) = -\beta c^2/r + \gamma c^2 r/2 \quad (3)$$

which may be fitted to data whenever the weak gravity limit is applicable.

The conformal theory thus not only generalizes Newton (Eq. (3)) it also generalizes Schwarzschild (Eq. (2)), and even does so in way which is then able to naturally recover both the Newton and Schwarzschild phenomenologies on the appropriate distance scales. Since the conformal theory recovers the requisite solutions to Einstein gravity on small enough distance scales (even while never recovering the Einstein Equations themselves - observation only demands the recovery of the solutions not of the equations), that fact alone makes the theory indistinguishable from and just as viable as the Einstein theory on those distance scales, something recognized by Eddington (1922) as far back as the very early days of Relativity. (Eddington was not aware of the full exact solution of Eq. (2) but was aware that it was a solution to fourth order gravity in the restricted case where  $\gamma = 0$ . It was only much later that the complete and exact solution of Eq. (2) was found and that its consistency was established by successfully matching it on to the associated exact interior solution (Mannheim and Kazanas 1994)). Thus in this sense conformal gravity should always have been considered as a viable explanation of solar system physics. That it never was so considered was in part due to the fact that strict conformal symmetry requires that all particles be massless, something which would appear to immediately rule the symmetry out. However, with the advent of modern spontaneously broken gauge theories manifest in the other three fundamental interactions, it is now apparent that mass can still be generated in the vacuum in otherwise dimensionless theories like the one associated with the action of Eq. (1). (And, interestingly, such dynamical mass generation is even found to still lead to geodesic motion (Mannheim 1993a), despite the fact that the associated mass generating Higgs scalar field which accompanies a test particle carries its own energy and momentum which the gravitational field also sees). Hence, it would appear that today the only non-relativistic way to distinguish between the two covariant theories is to explore their observational implications on larger distance scales where the linear potential term first makes itself manifest.

A first step towards this phenomenological end was taken recently by Mannheim (1993b) with the above non-relativistic potential  $V(r)$  being used in conjunction solely with observed surface brightness data (i.e. without assuming any dark matter) to fit the rotation curves of four representative galaxies. The particular choice of galaxies was guided by the recent comprehensive survey of the *H I* rotation curves of spiral galaxies made by Casertano and van Gorkom (1991) who found that those data fall into essentially four general groups characterized by specific correlations between the maximum rotation velocity and the luminosity.

In order of increasing luminosity the four groups are dwarf, intermediate, compact bright, and large bright galaxies. Thus one representative galaxy from each group was studied, respectively the galaxies DDO 154 (a gas dominated rather than star dominated galaxy), NGC 3198, NGC 2903, and NGC 5907, with the fitting of Mannheim (1993b) being reproduced here as Fig. (1). (The reader is referred to the original paper for details). For NGC 3198 the rotation curve of Begeman (1989) and the surface brightness data of Wevers et al. (1986) and Kent (1987) were used, for NGC 2903 the data were taken from Begeman (1987) and Wevers et al. (1986), for NGC 5907 from van Albada and Sancisi (1986) and Barnaby and Thronson (1992), and for DDO 154 from Carignan and Freeman (1988) and Carignan and Beaulieu (1989). (While Carignan and his coworkers favor a distance of 4 Mpc to DDO 154, Krumm and Burstein 1984 favor 10 Mpc. Since the gas contribution is extremely distance sensitive, for completeness we opted to fit this galaxy at both the candidate distances). As can be seen from Fig. (1), the conformal theory appears to be able to do justice to a data set which involves a broad range of luminosities, and to even do so without the need for dark matter, a point we analyze further below.

In order to apply the linear potential to an extended object such as a disk it was found helpful to develop a general formalism, with the results of the formalism being used in Mannheim (1993b) to produce the fits of Fig. (1). In Sec. (2) of the present paper we present the actual details of the derivation of the formalism (something that will be useful for future studies), with the formalism actually even being of interest in its own right since it extends to linear potentials the earlier work of Toomre (1963), Freeman (1970), and Casertano (1983) on Newtonian disks. In Sec. (3) we analyze some of the general systematics of galactic rotation curve fitting, and compare and contrast our fitting with that of the standard dark matter theory. Additionally in Sec. (3) we discuss the status of the Tully-Fisher relation (Tully and Fisher 1977) in conformal gravity, a relation which informs us that for regular spiral galaxies the velocity (i.e. the total gravitational potential) is universally normalized to the luminosity, a quantity which is fixed by the explicit amount of visible matter (i.e. that matter which is not non-luminous) in each of the relevant galaxies. The very existence of the Tully-Fisher relation thus quite strongly suggests that the total gravitational potential is fixed by the amount of luminous matter in a galaxy, and that galaxies should accordingly be predominantly luminous. Consequently, given the rotation curve data, it would appear that it is the rule for calculating the potential which therefore needs to be changed, to thus necessitate deviations from Newton on galactic distance scales such as those predicted in our theory. Now a theory with a linearly rising potential will lead to ever bigger deviations from Newton on ever bigger distance scales, a trend which in a first approximation nicely parallels that found in the standard theory where ever increasing amounts of dark matter are required as distance scales get larger. Consequently, in Sec. (4) we both develop an appropriate formalism for and make a first application of the conformal theory to the first available scale beyond galaxies, namely that of clusters of galaxies. Interestingly, we find that the conformal theory deviates from Newton there by just the amount needed to nicely accommodate a virialized cluster core without invoking dark matter, though the theory may turn out to have some difficulties should entire clusters prove to be virialized. In Sec. (5) we discuss the implications for gravitational theory of the recent round of microlensing searches for astrophysical dark matter, and show that the data presented so far leave the conformal gravity theory viable.

## (2) The Potential of an Extended Disk

In order to handle the weak gravity potential of an extended object such as a disk of stars each with gravitational potential  $V(r) = -\beta c^2/r + \gamma c^2 r/2$  many ways are possible, with perhaps the most popular being a method due to Toomre (1963) which was originally developed for thin Newtonian disks. Since that method does not immediately appear to generalize to linear potentials, we have instead generalized his approach first to non-thin Newtonian disks (a step also taken by Casertano 1983) and then to disks with linear potentials.

To determine the Newtonian potential of an axially symmetric (but not yet necessarily thin) distribution of matter sources with matter volume density function  $\rho(R, z')$  we need to evaluate the quantity

$$V_\beta(r, z) = -\beta c^2 \int_0^\infty dR \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' \frac{R\rho(R, z')}{(r^2 + R^2 - 2Rr\cos\phi' + (z - z')^2)^{1/2}} \quad (4)$$

where  $R, \phi', z'$  are cylindrical source coordinates and  $r$  and  $z$  are the only observation point coordinates of relevance. To evaluate Eq. (4) it is convenient to make use of the cylindrical Green's function Bessel function expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{m=-\infty}^\infty \int_0^\infty dk J_m(kr) J_m(kr') e^{im(\phi - \phi') - k|z - z'|} \quad (5)$$

whose validity can readily be checked by noting that use of the identity

$$\nabla^2 [J_m(kr) e^{im\phi - k|z - z'|}] = -2kJ_m(kr) e^{im\phi} \delta(z - z') \quad (6)$$

leads to the relation

$$\nabla^2 \left( \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}') \quad (7)$$

(In his original study Toomre used a Bessel function discontinuity formula (essentially Eq. (6)) which only appears to be applicable to thin disks. Using the full completeness properties of the Bessel functions enables us to treat non-thin disks as well). While Eq. (5) is standard, it is not utilized as often as the more familiar modified Bessel function expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{2}{\pi} \sum_{m=-\infty}^\infty \int_0^\infty dk \cos[k(z - z')] I_m(kr_{<}) K_m(kr_{>}) e^{im(\phi - \phi')} \quad (8)$$

since the product of the two modified Bessel functions has much better convergence properties at infinity than the product of the two ordinary Bessel functions. Nonetheless, the ordinary Bessel functions do actually vanish at infinity which is sufficient for our purposes here. A disadvantage of the expansion of Eq. (8) is that it involves oscillating  $z$  modes rather than the bounded  $z$  modes given in Eq. (5), with the bounded form of Eq. (5) actually being extremely convenient for a disk whose matter distribution is concentrated around  $z = 0$ . An additional shortcoming of the expansion of Eq. (8) is that when it is inserted into Eq. (4) it requires the  $R$  integration range to be broken up into two separate pieces at the point of observation. However, inserting Eq. (5) into Eq. (4) leads to

$$V_\beta(r, z) = -2\pi\beta c^2 \int_0^\infty dk \int_0^\infty dR \int_{-\infty}^\infty dz' R\rho(R, z') J_0(kr) J_0(kR) e^{-k|z - z'|} \quad (9)$$

which we see requires no such break up. Finally, taking the disk to be infinitesimally thin (viz.  $\rho(R, z') = \Sigma(R)\delta(z')$ ) then yields for points with  $z = 0$  the potential

$$V_\beta(r) = -2\pi\beta c^2 \int_0^\infty dk \int_0^\infty dR R \Sigma(R) J_0(kr) J_0(kR) \quad (10)$$

which we immediately recognize as Toomre's original result for an infinitesimally thin disk. In passing we note that Eq. (9) also holds for points which do not lie in the  $z = 0$  plane of the disk, and also applies to disks whose thickness may not in fact be negligible, with the form of Eq. (9) being particularly convenient if the fall-off of the matter distribution in the  $z$  direction is itself exponential (see below).

For our purposes here, the expansion of Eq. (5) can immediately be applied to the linear potential case too, and this leads directly (on setting  $|\mathbf{r} - \mathbf{r}'| = (\mathbf{r} - \mathbf{r}')^2/|\mathbf{r} - \mathbf{r}'|$ ) to the potential

$$\begin{aligned} V_\gamma(r, z) &= \frac{\gamma c^2}{2} \int_0^\infty dR \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' R \rho(R, z') [r^2 + R^2 - 2rR \cos\phi' + (z - z')^2]^{1/2} \\ &= \pi \gamma c^2 \int_0^\infty dk \int_0^\infty dR \int_{-\infty}^\infty dz' R \rho(R, z') [(r^2 + R^2 + (z - z')^2) J_0(kr) J_0(kR) \\ &\quad - 2rR J_1(kr) J_1(kR)] e^{-k|z - z'|} \end{aligned} \quad (11)$$

Equation (11) then reduces at  $z = 0$  for infinitesimally thin disks to the compact expression

$$V_\gamma(r) = \pi \gamma c^2 \int_0^\infty dk \int_0^\infty dR R \Sigma(R) [(r^2 + R^2) J_0(kr) J_0(kR) - 2rR J_1(kr) J_1(kR)] \quad (12)$$

If the  $k$  integrations are performed first in Eqs. (10) and (12) they lead to singular hypergeometric functions whose subsequent  $R$  integrations contain infinities which, even while they are in fact mild enough to be integrable (as long as  $\Sigma(R)$  is sufficiently damped at infinity), nonetheless require a little care when being carried out numerically. Thus unlike the sphere whose potential is manifestly finite at every step of the calculation, the disk, because of its lower dimensionality, actually encounters infinities at any interior point of observation on the way to a final finite answer. However, since the final answer is finite, it should be possible to obtain this answer without ever encountering any infinities at any stage of the calculation at all; and indeed, if the distribution function  $\Sigma(R)$  is available in a closed form, then performing the  $R$  integration before the  $k$  integration can yield a calculation which is finite at every stage. Thus, for the exponential disk

$$\Sigma(R) = \Sigma_0 e^{-\alpha R} \quad (13)$$

where  $1/\alpha = R_0$  is the scale length of the disk and  $N = 2\pi \Sigma_0 R_0^2$  is the number of stars in the disk, use of the standard Bessel function integral formulas

$$\int_0^\infty dR R J_0(kR) e^{-\alpha R} = \frac{\alpha}{(\alpha^2 + k^2)^{3/2}} \quad (14)$$

$$\int_0^\infty dk \frac{J_0(kr)}{(\alpha^2 + k^2)^{3/2}} = (r/2\alpha) [I_0(\alpha r/2) K_1(\alpha r/2) - I_1(\alpha r/2) K_0(\alpha r/2)] \quad (15)$$

then leads directly to Freeman's original result, viz.

$$\begin{aligned} V_\beta(r) &= -2\pi \beta c^2 \Sigma_0 \int_0^\infty dk \frac{\alpha J_0(kr)}{(\alpha^2 + k^2)^{3/2}} \\ &= -\pi \beta c^2 \Sigma_0 r [I_0(\alpha r/2) K_1(\alpha r/2) - I_1(\alpha r/2) K_0(\alpha r/2)] \end{aligned} \quad (16)$$

for the Newtonian potential of an exponential disk. The use of the additional integral formula

$$\int_0^\infty dR R^2 J_1(kR) e^{-\alpha R} = \frac{3\alpha k}{(\alpha^2 + k^2)^{5/2}} \quad (17)$$

and a little algebra (involving eliminating  $J_1(kr) = -(dJ_0(kr)/dk)/r$  via an integration by parts) enable us to obtain for the linear potential contribution the expression

$$V_\gamma(r) = \pi \gamma c^2 \Sigma_0 \int_0^\infty dk \left( \frac{\alpha r^2 J_0(kr)}{(\alpha^2 + k^2)^{3/2}} - \frac{9\alpha J_0(kr)}{(\alpha^2 + k^2)^{5/2}} + \frac{15\alpha^3 J_0(kr)}{(\alpha^2 + k^2)^{7/2}} - \frac{6\alpha k r J_1(kr)}{(\alpha^2 + k^2)^{5/2}} \right)$$

$$= \pi\gamma c^2 \Sigma_0 \int_0^\infty dk J_0(kr) \left( \frac{\alpha r^2}{(\alpha^2 + k^2)^{3/2}} + \frac{15\alpha}{(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^3}{(\alpha^2 + k^2)^{7/2}} \right) \quad (18)$$

Equation (18) is readily evaluated through use of the modified Bessel function recurrence relations

$$\begin{aligned} I'_0(z) &= I_1(z) \quad , \quad I'_1(z) = I_0(z) - I_1(z)/z \\ K'_0(z) &= -K_1(z) \quad , \quad K'_1(z) = -K_0(z) - K_1(z)/z \end{aligned} \quad (19)$$

in conjunction with Eq. (15) and its derivatives, and yields

$$\begin{aligned} V_\gamma(r) &= \pi\gamma c^2 \Sigma_0 \{ (r/\alpha^2) [I_0(\alpha r/2) K_1(\alpha r/2) - I_1(\alpha r/2) K_0(\alpha r/2)] \\ &\quad + (r^2/2\alpha) [I_0(\alpha r/2) K_0(\alpha r/2) + I_1(\alpha r/2) K_1(\alpha r/2)] \} \end{aligned} \quad (20)$$

To obtain test particle rotational velocities we need only differentiate Eqs. (16) and (20) with respect to  $r$ . This is readily achieved via repeated use of the recurrence relations of Eqs. (19) which form a closed set under differentiation so that higher modified Bessel functions such as  $I_2(\alpha r/2)$  and  $K_2(\alpha r/2)$  are not encountered; and the procedure is found to yield

$$\begin{aligned} rV'(r) &= (N\beta c^2 \alpha^3 r^2/2) [I_0(\alpha r/2) K_0(\alpha r/2) - I_1(\alpha r/2) K_1(\alpha r/2)] \\ &\quad + (N\gamma c^2 r^2 \alpha/2) I_1(\alpha r/2) K_1(\alpha r/2) \end{aligned} \quad (21)$$

Using the asymptotic properties of the modified Bessel functions we find that at distances much larger than the scale length  $R_0$  Eq. (21) yields

$$rV'(r) \rightarrow \frac{N\beta c^2}{r} + \frac{N\gamma c^2 r}{2} - \frac{3N\gamma c^2 R_0^2}{4r} \quad (22)$$

as would be expected. We recognize the asymptotic Newtonian term to be just  $N\beta c^2/r$  where  $N$  is the number of stars in the disk. The quantity  $N\beta c^2$  is usually identified as  $MG$  with  $M$  being taken to be the mass of the disk. For normalization purposes it is convenient to use this coefficient to define the velocity  $v_0 = c(N\beta/R_0)^{1/2}$ , the velocity that a test particle would have if orbiting a Newtonian point galaxy with the same total mass at a distance of one scale length. In terms of the convenient dimensionless parameter  $\eta = \gamma R_0^2/\beta$  Eq. (21) then yields for the rotational velocity  $v(r)$  of a circular orbit in the plane of a thin exponential disk the exact expression

$$v^2(r)/v_0^2 = (r^2 \alpha^2/2) [I_0(\alpha r/2) K_0(\alpha r/2) + (\eta - 1) I_1(\alpha r/2) K_1(\alpha r/2)] \quad (23)$$

an expression which is surprisingly compact. For thin disks then all departures from the standard Freeman result are thus embodied in the one parameter  $\eta$  in the simple manner indicated.

Beyond making actual applications to galaxies, a further advantage of having an exact solution in a particular case is that it can be used to test a direct numerical evaluation of the galactic potential (which involves integrable infinities) by also running the program for a model exponential disk. Also, it is possible to perform the calculation analytically in various other specific cases. For a thin axisymmetric disk with a Gaussian surface matter distribution  $\Sigma(R) = \Sigma_0 \exp(-\alpha^2 R^2)$  and  $N = \pi \Sigma_0 / \alpha^2$  stars (this being a possible model for the sometimes steeper central region of a galaxy in cases where there may be no spherical bulge) we find for the complete rotational velocity the expression

$$rV'(r) = \pi^{1/2} N \beta c^2 \alpha^3 r^2 [I_0(\alpha^2 r^2/2) - I_1(\alpha^2 r^2/2)] e^{-\alpha^2 r^2/2}$$

$$+(\pi^{1/2}N\gamma c^2\alpha r^2/4)[I_0(\alpha^2 r^2/2) + I_1(\alpha^2 r^2/2)]e^{-\alpha^2 r^2/2} \quad (24)$$

Similarly, for a spherically symmetric matter distribution (such as the central bulge region of a galaxy) with radial matter density  $\sigma(r)$  and  $N = 4\pi \int dr' r'^2 \sigma(r')$  stars a straightforward calculation yields

$$rV'(r) = \frac{4\pi\beta c^2}{r} \int_0^r dr' \sigma(r') r'^2 + \frac{2\pi\gamma c^2}{3r} \int_0^r dr' \sigma(r') (3r^2 r'^2 - r'^4) + \frac{4\pi\gamma c^2 r^2}{3} \int_r^\infty dr' \sigma(r') r' \quad (25)$$

which can readily be integrated once a particular  $\sigma(r)$  is specified. For spherical systems  $\sigma(r)$  is not actually measured directly. Rather, it is extracted from the surface matter distribution  $I(R)$  via an Abel transform

$$\sigma(r) = -\frac{1}{\pi} \int_r^\infty dR \frac{I'(R)}{(R^2 - r^2)^{1/2}} \quad , \quad I(R) = 2 \int_R^\infty dr \frac{\sigma(r)r}{(r^2 - R^2)^{1/2}} \quad (26)$$

thus leading to double integrals in Eq. (25). However, reduction of these integrals to one-dimensional integrals over  $I(R)$  is possible since we can rewrite Eq. (25) as

$$rV'(r) = -2\beta c^2 S(r) + \frac{2\beta c^2}{r} \int_0^r dr' S(r') + \gamma c^2 r \int_0^r dr' S(r') - \frac{\gamma c^2}{r} \int_0^r dr' r'^2 S(r') \quad (27)$$

where the strip brightness  $S(x)$  obeys

$$S(x) = 2 \int_x^\infty dR \frac{RI(R)}{(R^2 - x^2)^{1/2}} \quad , \quad \sigma(x) = -\frac{S'(x)}{2\pi x} \quad (28)$$

Noting that

$$\begin{aligned} \frac{d}{dr} \left\{ 2 \int_r^\infty dR RI(R) \arcsin\left(\frac{r}{R}\right) \right\} &= -\pi r I(r) + S(r) \quad , \\ \frac{d}{dr} \left\{ 2 \int_r^\infty dR RI(R) \left[ R^2 \arcsin\left(\frac{r}{R}\right) - r(R^2 - r^2)^{1/2} \right] \right\} &= -\pi r^3 I(r) + 2r^2 S(r) \end{aligned} \quad (29)$$

enables us to conveniently reexpress Eq. (25) directly in terms of  $I(R)$ , viz.

$$\begin{aligned} rV'(r) &= \frac{4\beta c^2}{r} \int_r^\infty dR RI(R) \left[ \arcsin\left(\frac{r}{R}\right) - \frac{r}{(R^2 - r^2)^{1/2}} \right] + \\ &\quad \frac{2\pi\beta c^2}{r} \int_0^r dR RI(R) + \frac{\gamma c^2 \pi}{2r} \int_0^r dR RI(R) (2r^2 - R^2) \\ &\quad + \frac{\gamma c^2}{r} \int_r^\infty dR RI(R) \left[ (2r^2 - R^2) \arcsin\left(\frac{r}{R}\right) + r(R^2 - r^2)^{1/2} \right] \end{aligned} \quad (30)$$

with the Newtonian contribution having previously been noted by Kent (1986).

Beyond the exact expressions obtained above there is one other case of practical interest namely that of non-thin but separable disks, a case which can also be greatly simplified by our formalism. For such separable disks we set  $\rho(R, z') = \Sigma(R)f(z')$  where the usually symmetric thickness function  $f(z') = f(-z')$  is normalized according to

$$\int_{-\infty}^\infty dz' f(z') = 2 \int_0^\infty dz' f(z') = 1 \quad (31)$$

Recalling that

$$e^{-k|z-z'|} = \theta(z-z')e^{-k(z-z')} + \theta(z'-z)e^{+k(z-z')} \quad (32)$$



we find that Eqs. (9) and (11) then yield for points with  $z = 0$

$$V_\beta(r) = -4\pi\beta c^2 \int_0^\infty dk \int_0^\infty dR \int_0^\infty dz' R \Sigma(R) f(z') J_0(kr) J_0(kR) e^{-kz'} \quad (33)$$

and

$$V_\gamma(r) = 2\pi\gamma c^2 \int_0^\infty dk \int_0^\infty dR \int_0^\infty dz' R \Sigma(R) f(z') \times [(r^2 + R^2 + z'^2) J_0(kr) J_0(kR) - 2rR J_1(kr) J_1(kR)] e^{-kz'} \quad (34)$$

in the separable case. Further simplification is possible if the radial dependence is again exponential (viz.  $\Sigma(R) = \Sigma_0 \exp(-\alpha R)$ ) and yields, following some algebra involving the use of the recurrence relation  $J'_1(z) = J_0(z) - J_1(z)/z$ , the expressions

$$rV'_\beta(r) = 2N\beta c^2 \alpha^3 r \int_0^\infty dk \int_0^\infty dz' \frac{f(z') e^{-kz'} k J_1(kr)}{(\alpha^2 + k^2)^{3/2}} \quad (35)$$

and

$$rV'_\gamma(r) = N\gamma c^2 \alpha^3 r \int_0^\infty dk \int_0^\infty dz' f(z') e^{-kz'} \times \left( -\frac{4r J_0(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{6\alpha^2 r J_0(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{(r^2 + z'^2) k J_1(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{9k J_1(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^2 k J_1(kr)}{(\alpha^2 + k^2)^{7/2}} \right) \quad (36)$$

As regards actual specific forms for  $f(z')$ , two particular ones have been identified via the surface photometry of edge on galaxies, one by van der Kruit and Searle (1981), and the other by Barnaby and Thronson (1992). Respectively they are

$$f(z') = \text{sech}^2(z'/z_0)/2z_0 \quad (37)$$

and

$$f(z') = \text{sech}(z'/z_0)/\pi z_0 \quad (38)$$

each with appropriate scale height  $z_0$ . (A recent faint starlight search by Sackett et. al. 1994 suggests that for NGC 5907 the above previously detected exponential drop gradually softens into a power-law behavior at larger scale heights. The effect of this tail on fitting will be negligible if it possesses the same mass to light ratio as all the other matter that had previously been detected in the galaxy, and so we shall not consider it further here). We note that both of the thickness functions of Eqs. (37) and (38) are falling off very rapidly in the  $z'$  direction just like the Bessel function expansion itself of Eq. (5). Consequently, Eqs. (33) - (36) will now have very good convergence properties. The thickness function of Eq. (37) is found to lead to rotational velocities of the form

$$rV'_\beta(r) = (N\beta c^2 \alpha^3 r^2/2) [I_0(\alpha r/2) K_0(\alpha r/2) - I_1(\alpha r/2) K_1(\alpha r/2)] - N\beta c^2 \alpha^3 r \int_0^\infty dk \frac{k^2 J_1(kr) z_0 \beta (1 + k z_0/2)}{(\alpha^2 + k^2)^{3/2}} \quad (39)$$

where

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{(1+t)} \quad (40)$$

and

$$rV'_\gamma(r) = N\gamma c^2 \alpha^3 r \int_0^\infty dk (1 - k z_0 \beta(1 + k z_0/2))$$

$$\begin{aligned} & \times \left( -\frac{2rJ_0(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{3\alpha^2 rJ_0(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{r^2 kJ_1(kr)}{2(\alpha^2 + k^2)^{3/2}} + \frac{9kJ_1(kr)}{2(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^2 kJ_1(kr)}{2(\alpha^2 + k^2)^{7/2}} \right) \\ & + N\gamma c^2 \alpha^3 r \int_0^\infty dk \frac{kJ_1(kr)}{2(\alpha^2 + k^2)^{3/2}} \frac{d^2}{dk^2} \left( kz_0 \beta \left( 1 + \frac{kz_0}{2} \right) \right) \end{aligned} \quad (41)$$

Similarly, the thickness function of Eq. (38) leads to

$$rV'_\beta(r) = \frac{2N\beta c^2 \alpha^3 r}{\pi} \int_0^\infty dk \frac{kJ_1(kr)\beta(1/2 + kz_0/2)}{(\alpha^2 + k^2)^{3/2}} \quad (42)$$

and

$$\begin{aligned} rV'_\gamma(r) &= \frac{N\gamma c^2 \alpha^3 r}{\pi} \int_0^\infty dk \beta(1/2 + kz_0/2) \\ & \times \left( -\frac{4rJ_0(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{6\alpha^2 rJ_0(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{r^2 kJ_1(kr)}{(\alpha^2 + k^2)^{3/2}} + \frac{9kJ_1(kr)}{(\alpha^2 + k^2)^{5/2}} - \frac{15\alpha^2 kJ_1(kr)}{(\alpha^2 + k^2)^{7/2}} \right) \\ & - \frac{N\gamma c^2 \alpha^3 r}{\pi} \int_0^\infty dk \frac{kJ_1(kr)}{(\alpha^2 + k^2)^{3/2}} \frac{d^2}{dk^2} \left( \beta \left( \frac{1 + kz_0}{2} \right) \right) \end{aligned} \quad (43)$$

The great utility of these expressions is that all of the functions of  $\beta(x)$  and their derivatives which appear in Eqs. (39) - (43) converge very rapidly to their asymptotic values as their arguments increase. Consequently the  $k$  integrations in Eqs. (39) - (43) converge very rapidly numerically while encountering no singularities at all.

As a practical matter, the observed scale heights  $z_0$  are usually much smaller than any observed scale lengths  $R_0$ . Consequently the thickness corrections of Eqs. (39) - (43) usually only modify the thin disk formula of Eq. (21) in the central galactic region, and thus have essentially no effect on the linear potential contribution. For the Newtonian term the corrections of Eqs. (39) and (42) to the Freeman formula tend to reduce the overall Newtonian contribution (c.f. the sign of the integral term in Eq. (39) and Casertano 1983) and serve to ensure that the inner rotation curves of Fig. (1) are well described (see Mannheim 1993b) by the luminous Newtonian contribution, to thus clear the way to explore the effect of the linear term on the outer region of the rotation curve, a region where its presence is significant and where the thin disk formula of Eqs. (21) and (23) provides a very good approximation to the dynamics.

### (3) Some General Systematics of Galactic Rotation Curve Fitting

In order to understand the general features of the rotation curves of Fig. (1) it is instructive to consider the generic implications of the thin disk formula of Eq. (23), a two parameter formula with  $v_0$  fixing the overall normalization and  $\eta$  the relative contributions of the Newtonian and linear pieces. Moreover, if this (per galaxy) overall normalization is fixed by the peak in the rise of the inner rotation curve (the so called maximum disk fit in which the Newtonian disk contribution gets to be as large as it possibly can be), then essentially the entire shape of the rest of the curve is fixed by just the one parameter (per galaxy)  $\eta$ . As regards this maximum disk contribution, we note that the Newtonian term in Eq. (23) peaks at  $2.15R_0$  with  $v^2/v_0^2$  receiving a Newtonian contribution of 0.387. This Newtonian contribution comes down to half of this value (i.e. 0.194) at  $6.03R_0$ . Since the linear contribution is essentially negligible at  $2.15R_0$  (especially after we take the square root to get the velocity itself), if we choose the linear contribution at  $6.03R_0$  to be equal to the Newtonian contribution at that same distance (i.e. if numerically we set  $\eta$  equal to a critical value of 0.067, to thus fix our one free parameter (per galaxy) once and for all), we will then have essentially achieved flatness over the entire 2 to 6 scale length region. Now at 6 scale lengths both the Newtonian and linear terms are quickly approaching the asymptotic values exhibited in Eq. (22), and at close to 12 scale lengths (precisely at  $11.62R_0$ ) the linear term contribution to  $v^2/v_0^2$  is just 0.387, the original maximum disk value at

$2.15R_0$ . Consequently, between 6 and 12 scale lengths the rotation curve will again show little deviation from flatness without any further adjustment of parameters at all. However since the Newtonian contribution at 12 scale lengths is slightly bigger than the linear contribution at 2 scale lengths, the net outcome is that by 12 scale lengths the rotation curve is actually beginning to show a slight rise, with flatness only being achieved out to about 10 scale lengths. Thus in general we see that by varying just one parameter we can naturally achieve flatness over the entire 2 to 10 scale length region, this intriguingly being about as large a range of scale lengths as has up till now been observed in any rotation curve. In order to see just how flat a rotation curve it is in principle possible to obtain, we have varied  $\eta$  as a free parameter. Our most favored generic case is then obtained when  $\eta$  takes the value 0.069 (i.e. essentially the critical value), with the resulting generic rotational velocity curve being plotted in Fig. (2). Over the range from 3 to 10 scale lengths the ratio  $v(r)/v_0$  is found to take the values (0.666, 0.648, 0.632, 0.626, 0.628, 0.637, 0.651, 0.667) in unit step increases. Thus it has a spread of only  $\pm 3\%$  about a central value of 0.647 in this region. Additionally, we find that even at 15 scale lengths the ratio  $v(r)/v_0$  has still only increased to 0.763, a 14% increase over its value at 10 scale lengths. In the upper diagram in Fig. (2) we have plotted the generic  $\eta = 0.069$  rotation curve out to 10 scale lengths to show just how flat it can be. In the lower diagram in Fig. (2) we have shown the continuation out to 15 scale lengths where the ultimate asymptotic rise is becoming apparent. We have deliberately juxtaposed the two diagrams in Fig. (2) since the flatness out to 10 scale lengths is usually taken as being indicative of asymptotic flatness as well, with such ultimate flatness being characteristic (and even a primary motivation) of both isothermal gas sphere dark matter models and the MOND alternative (Milgrom 1983a, b, c). The possibility that flatness is only an intermediate and not an asymptotic phenomenon is one of the most unusual and distinctive features of the conformal gravity theory. (Of course it is always possible to build dark matter models with non flat asymptotic properties (see e.g. van Albada et al. 1985) since the dark matter theory is currently so unconstrained. However, our point here is that the conformal theory is the first theory in which rotation curves are actually required to ultimately rise, even being predicted to do so in advance of any data). As regards other possible values for  $\eta$ , if  $\eta$  exceeds the critical value of 0.067, then the curve will be flat for fewer scale lengths with the rise setting in earlier (given the large value for  $R_0$ , and hence  $\eta$ , that the galaxy UGC 2885 happens to possess (see below), we note in passing that a study of its outer region might thus provide a good opportunity to detect such a possible rise); while if  $\eta$  is less than the critical value, the curve will drop perceptibly before coming back to its maximum disk value at a greater distance.

As regards the generic critical value for  $\eta$ , we note that for a typical galaxy with a mass of  $10^{11}$  solar masses and a 3 kpc scale length, the required value for the galactic  $\gamma_{gal}$  ( $= N\gamma_{star}$  where  $\gamma_{star}$  is the typical  $\gamma$  used in the stellar potential  $V(r)$  of Eq. (3)) then turns out to be of order  $10^{-29}/\text{cm}$ , which, intriguingly, is of order the inverse Hubble radius. Moreover, this characteristic value is in fact numerically attained in the fits of Fig. (1) for the stellar disk contribution in all of our four chosen galaxies (viz.  $\gamma(154)=2.5\times 10^{-30}/\text{cm}$ ,  $\gamma(3198)=3.5\times 10^{-30}/\text{cm}$ ,  $\gamma(2903)=7.6\times 10^{-30}/\text{cm}$ ,  $\gamma(5907)=5.7\times 10^{-30}/\text{cm}$ ). (While this same cosmological value is also found for DDO 154, in some other aspects (such as possessing a rotation curve which has no observed flat region at all) the fitting to this dwarf irregular is found to be anomalous (see Mannheim 1993b) presumably because the galaxy is gas rather than star dominated. Hence we shall only regard the three other galaxies, all regular spirals, as typical for the purposes of our discussion here). Thus not only is  $\gamma_{gal}$  making the observed representative curves flat in the observed region, and not only is it doing so with an effectively universal value, it is doing so with a value which is already known to be of astrophysical significance; thereby suggesting that  $\gamma_{gal}$  may be of cosmological origin, perhaps being related to the scale at which galaxies fluctuate out of the cosmological background.

Additionally, we note that this apparent universality for  $\gamma_{gal}$  has implications for the status of the Tully-

Fisher relation in our theory. Specifically, the average velocity  $v_{ave}$  of the flat part of the critical generic curve (but only of the flat part since the curve must ultimately rise in our theory) is equal to the maximum disk value since the curve is flat in that region. Thus we can set (using  $N = 2\pi\Sigma_0 R_0^2$  and letting  $L$  denote the galactic luminosity)

$$v_{ave}^4 = \left( \frac{0.387N\beta c^2}{R_0} \right)^2 = 0.300\pi\Sigma_0\beta^2 c^4 L \left( \frac{N}{L} \right) \quad (44)$$

At the critical value for  $\eta$  (the fits yield  $\eta(3198)=0.044$ ,  $\eta(2903)=0.038$ ,  $\eta(5907)=0.057$ ) we also can set

$$\gamma_{gal}c^2 = \frac{0.067N\beta c^2}{R_0^2} = 0.134\pi\Sigma_0\beta c^2 \quad (45)$$

so that Eq. (44) may be rewritten as

$$v_{ave}^4 = 2.239\gamma_{gal}\beta c^4 L \left( \frac{N}{L} \right) \quad (46)$$

If we assume that all galaxies possess the same universal value for the disk mass to light ratio (our fits yield  $M/L(3198)=4.2$ ,  $M/L(2903)=3.5$ ,  $M/L(5907)=6.1$  in units of  $M_\odot/L_{B\odot}$ ), we then see that given a universal  $\gamma_{gal}$  and a universal  $\eta$ , Eq. (46) then yields noneother than the Tully-Fisher velocity-luminosity relation. (Observationally the Tully-Fisher relation is not thought to hold for the stellar component of the dwarf irregular DDO 154, as may be anticipated since DDO 154 is phenomenologically found to have an anomalously small  $M/L$  ratio ( $M/L(154)$  takes the value 1.4 in our fits and is essentially zero in the dark matter and MOND fits of Begeman, Broeils, and Sanders 1991) - since the above discussion does not include any non-stellar component it is anyway not applicable to gas dominated galaxies). Additionally, according to Eq. (45) the universality of  $\gamma_{gal}$  also entails the universality of  $\Sigma_0$ , the central surface brightness, an as yet unexplained phenomenological feature first identified for spirals by Freeman (1970). (In turn the universality of  $\Sigma_0$  entails a mass - radius squared relation for galaxies). The (near) universality of  $\gamma_{gal}$  and of  $\eta$  (the near universality found for  $\eta$  is accounted for by the phenomenological fact that the scale lengths  $R_0$  of the three spiral galaxies in our sample are all quite close to each other) thus correlates in one fell swoop the observed flatness of rotation curves, the universality of  $\Sigma_0$ , and the Tully-Fisher relation, and does so in a theory in which rotation curves must eventually rise. As such, the above given discussion provides a generalization to axially symmetric systems of an earlier discussion (Kazanas 1991, Kazanas and Mannheim 1991b) based on the simplification of using Eq. (2) itself as the galactic metric. As we now see, the ideas developed in those two papers carry over to the present more detailed treatment. (In passing we should point out the mass - radius squared relation which was also identified in those two previous papers was actually found to have phenomenological validity on many other astrophysical scales as well, something which still awaits an explanation).

It is important to note that the above discussion does not constitute a complete first principles derivation of the Tully-Fisher relation. Rather, even while doing so in an unforced and natural way, the discussion nonetheless takes advantage of the phenomenological facts that  $\Sigma_0$  and  $R_0$  are each quite close to universal without explaining why this is so. (Universal  $\Sigma_0$  entails Tully-Fisher for the maximum disk peak velocity in galaxies which have no significant inner region bulge, while universal  $\eta$  then extends the Tully-Fisher relation to the average rotation velocity to the extent that the curve is in fact flat.) However, since Eq. (45) does correlate  $\Sigma_0$  with  $\gamma_{gal}$  and since  $\gamma_{gal}$  is numerically of cosmological significance, our analysis does suggest that the theoretical establishing of a cosmological origin for  $\gamma_{gal}$  in which it would emerge as being of order the inverse Hubble radius (presumably in a theory of galaxy formation via cosmological inhomogeneities) would then lead naturally to intermediate region flatness, to the Tully-Fisher relation, and to the universality

of  $\Sigma_0$ . As regards the Tully-Fisher relation in general, we note that it actually has two aspects to it, one being the obvious fact that it is the fourth power (as opposed to any other possible power) of the velocity which is universally related to the luminosity, and the other being the much deeper fact that there is actually a universal correlation at all, i.e. that the velocity (which is fixed by the total gravitational potential) is correlated with the luminosity (which is fixed by the visible matter alone) rather than being correlated with any possible non-luminosity. Thus the very fact that there is a Tully-Fisher relation at all thus quite strongly suggests that galaxies should therefore be predominantly luminous, and that it is the rule for calculating the potential which hence needs to be changed. Moreover, our ability to obtain Eqs. (44-46) and the fits of Fig. (1) in our theory follows precisely because the linear potential is integrated over the same luminous matter distribution as the Newtonian potential, to thus automatically normalize both the contributions (and hence the total velocity) to one and the same luminosity in a completely natural manner.

While we have categorized our fits as having two parameters per galaxy, the actual situation is slightly more constrained. Specifically, we note that the Newtonian and linear contributions are both proportional to  $N$  according to Eq. (22). Thus if there existed universal average stellar parameters  $\beta$  and  $\gamma$  to serve as input for Eqs. (4) and (11),  $\eta$  would then be fixed by the scale length  $R_0$  of each galaxy, resulting in one parameter ( $N$ ) per galaxy fits. Ordinarily, one thinks of  $\beta$  as being the Schwarzschild radius of the Sun, and then in the fits the numerical value of the mass to light ratio of the galaxy is allowed to vary freely in the fitting, with disk  $M/L$  ratios then being found which are actually remarkably close to each other (without such closeness there would be severe violations of the Tully-Fisher relation because of the  $N/L = M/LM_\odot$  factor in Eqs. (44) and (46)). However, in reality each galaxy comes with its own particular mix of stars, both in overall population and, even more significantly, in the spatial distribution of the mix. Now, of course ideally we should integrate Eq. (4) over the true stellar distribution allowing  $\beta$  to vary with position according to where the light and heavy stars (stars whose luminosities do not simply scale linearly with their masses) are physically located within the stellar disk. Instead we use an average  $\beta$  (which incidentally enables us to derive exact formulas such as Eq. (9)). However, two galaxies with the identical morphological mix of stars but with different spatial distributions of those stars should each be approximated by a different average  $\beta$ , since the Newtonian potential weights different distances unequally. Since we do not give two galaxies of this type different average  $\beta$  parameters to begin with, we can then compensate later by giving them different mass to light ratios (even though for this particular example we gave them the same morphological mix). Hence we extract out a quantity  $N\beta_\odot$  from the data which simulates  $N_{ave}\beta_{ave}$  where  $N_{ave}$  is the true average number of stars in the galaxy. Because of the difference between these two ways of defining the number of stars in a galaxy, it is not clear whether the currently quoted mass to light ratios as found in the fits (in essentially all theories of rotation curve systematics) are merely reflecting this difference or whether they are exploiting this uncertainty to come up with possibly unwarrantable mass to light ratios. Thus a first principles determination of actual values or of a range of allowed values of galactic disk mass to light ratios prior to fitting would be extremely desirable, with (as we shall see below) the recent round of microlensing observations actually taking a first step in this direction.

A precisely similar situation also obtains for the  $\gamma$  dependent terms. Again we use an average stellar  $\gamma$  and compensate for its possible average variation from galaxy to galaxy by allowing the galactic disk gamma to light ratio ( $\gamma_{gal}/L = N\gamma/L$ ), and hence the galactic  $\eta$ , to vary phenomenologically (i.e. we use  $N\gamma$  to simulate  $N_{ave}\gamma_{ave}$  where  $N$  is determined once and for all by normalizing the data to  $N\beta_\odot$ ). The fits to our representative galaxies are found to yield  $N\gamma/L_B(3198)=3.9$ ,  $N\gamma/L_B(2903)=5.1$ ,  $N\gamma/L_B(5907)=3.2$  (in units of  $10^{-40}/\text{cm}/L_{B\odot}$ ), values which again are remarkably close to each other and which are of a par with the mass to light ratios  $M/L(3198)=4.2$ ,  $M/L(2903)=3.5$ ,  $M/L(5907)=6.1$  found for the same galaxies. We would not expect the  $M/\gamma_{gal}$  ratio to be the same for the entire sample, simply because even if the

stellar  $\beta$  and  $\gamma$  parameters were to change by the same proportion in going from one morphological type of star to another (a reasonable enough expectation), nonetheless, as the galactic spatial distributions change, the inferred average stellar  $\beta$  and  $\gamma$  parameters would then change in essentially unrelated ways, since the Newtonian and linear potentials weight the differing spatial regions of the galaxy quite differently to thus yield different average values. Nonetheless, it is intriguing to find that the variation in the average  $\beta$  and  $\gamma$  shows such mild dependence on specific galaxy within our sample;  $N$  and  $N_{ave}$  thus appear to be very close, with this small variation absorbing the variation in scale length  $R_0$  used to determine the various  $\eta$ . To within this (mild) variation, our fits are thus effectively one parameter per galaxy fits. (In its pure form the MOND explanation of the systematics of galactic rotation curves is also a one parameter per galaxy theory. However, in its successful practical applications (Begeman, Broeils, and Sanders 1991), it is generally found necessary to introduce at least one more fitting parameter per galaxy, such as by allowing a (generally quite mild) variation in the fundamental acceleration parameter  $a_0$  over the galactic sample. Phenomenologically then MOND would thus appear to be on a par with our linear potential theory). For our linear potential theory we note that given the apparent uniformity of the average stellar  $\gamma/\beta$  ratio, we see that we really have to normalize  $N$  to the maximum disk mass and that we are really not free to vary the normalizations of the Newtonian and linear pieces separately, since they both are proportional to  $N$ . Specifically, if we make the Newtonian piece too small we would have to arbitrarily increase the linear contribution, something we are not able to do in a consistent manner. Thus the Newtonian contribution in our fit cannot be too small. Similarly, it can never be allowed to be too large (this would give too high a velocity); and, hence, the Newtonian contribution in our theory is bounded both above and below, and essentially forced to the maximum disk mass; and thus our theory is reduced to almost parameter free fitting. Since dark matter fits can generally adjust the relative strengths of the luminous and dark matter pieces at will, they are not so constrained, and often yield much smaller luminous Newtonian contributions, and thus large amounts of dark matter, particularly in fits to dwarf galaxies. Thus a first principles determination of galactic disk mass to light ratios might enable one to discriminate between rival theories. (As we discuss below this is precisely beginning to happen via the microlensing observations, with the initial data apparently even supporting the conformal gravity maximum disk, minimal halo scenario.)

In order to compare our work with that of other approaches it is useful to clarify the significance of the term ‘flat rotation curve’. In the literature it is generally thought that rotation curves will be flat asymptotically (though of course the most significant aspect of the data is the fact that the curves deviate from the luminous Newtonian prediction at all, rather than in what particular way); and of course since our model predicts that velocities will eventually grow as  $r^{1/2}$ , the initial expectation is that our model is immediately ruled out. However, the rotation curve fits that have so far been made are not in fact asymptotic ones. Firstly, the *HII* optical studies pioneered by Rubin and coworkers (Rubin et al. 1970, 1978, 1980, 1982, 1985), even while they were indeed yielding flat rotation curves, were restricted to the somewhat closer in optical disk region since the *HII* regions are only to be found in the vicinity of hot stars which ionize those regions. And eventually, after a concentrated effort to carefully measure the surface brightnesses of such galaxies, it gradually became apparent (see e.g. Kalnajs 1983 and Kent 1986) that the *HII* curves could be described by a standard luminous Newtonian prediction after all; even in fact for galaxies such as UGC 2885 for which the data go out to as much as 80 kpc, a distance which turns out to only be of order 4 scale lengths ( $R_0=22$  kpc for UGC 2885, an atypically high value - this galaxy is just very big.) Thus, not only are the optical studies limited (by their very nature in fact) to the optical disk region where there is some detectable surface brightness, but, albeit by coincidence, it turns out that they are also limited to the region where an extended Newtonian source is actually yielding flat rotation curves to a rather good degree. Thus this inner region flatness has nothing at all to do with any possible asymptotic flatness, though it will

enable flatness to set in as early as 2 or 3 scale lengths in fits to any data which do go out to many more scale lengths.

While the *HII* data do not show any substantive non-canonical behavior, nonetheless, the pioneering work of Rubin and coworkers brought the whole issue of galactic rotation curves into prominence and stimulated a great deal of study in the field. Now it turns out that neutral hydrogen gas is distributed in galaxies out to much farther distances than the stars, thus making the *HI* studies ideal probes of the outer reaches of the rotation curves and of the luminous Newtonian prediction. (That *HI* studies might lead to a conflict with the luminous Newtonian prediction was noted very early by Freeman 1970 from an analysis of NGC 300 and M33, by Roberts and Whitehurst 1975 from an analysis of M31, and by Bosma 1978, 1981 who made the first large 21 cm line survey of spiral galaxies). Thus with the *HI* studies (there are now about 30 well studied cases) it became clear that there really was a problem with the interpretation of galactic rotation curve data, which the community immediately sought to explain by the introduction of galactic dark matter since the Newton-Einstein theory was presumed to be beyond question. (So much so in fact that Ostriker and Peebles 1973 had already introduced a spherical dark matter halo to stabilize otherwise unstable Newtonian disk galaxies). Fits to the *HI* data have been obtained using dark matter (Kent 1987 provides a very complete analysis), and while the fits are certainly phenomenologically acceptable, they nonetheless possess certain shortcomings. Far and away their most serious shortcoming is their ad hoc nature, with any found Newtonian shortfall then being retroactively fitted by an appropriately chosen dark matter distribution. In this sense dark matter is not a predictive theory at all but only a parametrization of the difference between observation and the luminous Newtonian expectation. As to possible dark matter distributions, no specific distribution, or explicit set of numerical parameters for a distribution, has convincingly been derived from first principles as a consequence, say, of galactic dynamics or formation theory (for a recent critical review see Sanders 1990). (The general community would not appear to regard any specific derivation as being all that convincing since no distribution has been heralded as being so theoretically secure that any failure of the data to conform to it would obligate the community to have to abandon the Newton-Einstein theory). Amongst the candidate dark matter distributions which have been considered in the literature the most popular is the distribution associated with a modified isothermal gas sphere (a two parameter spherical matter volume density distribution  $\rho(r) = \rho_0/(r^2 + r_0^2)$  with an overall scale  $\rho_0$  and an arbitrarily introduced non-zero core radius  $r_0$  which would cause dark matter to predominate in the outer rather than the inner region - even though a true isothermal sphere would have zero core radius). The appeal of the isothermal gas sphere is that it leads to an asymptotically logarithmic galactic potential and hence to asymptotically flat rotation curves, i.e. it is motivated by no less than the very data that it is trying to explain. However, careful analysis of the explicit dark matter fits is instructive. Recalling that the inner region (around, say,  $2R_0$  for definitiveness) is already flat for Newtonian reasons, the dark matter parameters are then adjusted so as to join on to this Newtonian piece (hence the ad hoc core radius  $r_0$ ) to give a continuously flat curve in the observed region, rather than one which either rises or falls to its presupposed eventual asymptotic flat value. This matching of the luminous and dark matter pieces is for the moment completely fortuitous (van Albada and Sancisi 1986 have even referred to it as a conspiracy) and not yet explained by galactic dynamics, even though it is only by such (assumed universal) matching galaxy by galaxy of the inner region Newtonian peak velocity to the presupposed constant asymptotic velocity that the dark matter models can achieve compatibility with the universal Tully-Fisher relation, with this treatment of the Tully-Fisher relation standing in sharp contrast to that provided by the conformal gravity theory which was discussed above. What is done in the dark matter fits is actually even a double conspiracy. Not only are the outer ( $10R_0$ ) and the inner ( $2R_0$ ) regions given the same velocity (by adjusting  $\rho_0$ ), the intermediate ( $6R_0$ ) region is adjusted through the core radius  $r_0$  to ensure that the curve does not fall and then rise again in that region. Hence flatness in

the  $r_0$  dominated region has almost nothing at all to do with the presumed asymptotically flat isothermal gas sphere contribution. Even worse, in the actual fits the dark matter contributions are found to actually still be rising at the largest observed ( $10R_0$ ) distances, and thus not yet taking on their asymptotic values at all. Hence the curves are made flat not by a flat dark matter contribution but rather by an interplay between a rising dark matter piece and a falling Newtonian one (an effect which is completely mirrored in the linear potential theory fits of Fig. (1)), with the asymptotically flat expectation not yet actually having even been tested. (Prospects for pushing the data out to farther distances are not good because *HI* surface densities typically fall off exponentially fast at the edge of the explored region.) Thus for the moment, even though both available *HI* and *HII* type data sets are flat in their respective domains, each data set is flat for its own coincidental reason, and it would appear to us that the region of true galactic asymptotics has yet to be explored; with the observed flatness of the galactic rotation curves (just like the apparent flatness of total proton proton scattering cross sections over many energy decades before an eventual rise) perhaps only being an intermediate rather than an asymptotic phenomenon.

As regards the conformal gravity theory fits, we see that is not in fact necessary to demand flatness in the asymptotic region in order to obtain flat rotation curves in the explored intermediate region. Thus, unlike dark matter fitting, we do not need to know the structure of the data prior to the fitting (when Mannheim and Kazanas first set out to analyze conformal gravity they had no idea what the fits might eventually look like at all), or need to adapt the model to any presupposed asymptotic flatness. Our linear potential theory is also more motivated theoretically than the dark matter models since Eq. (3) arises in a fundamental, fully covariant, uniquely specified theory, while, despite all the attention it has been given, the dark matter spherical halo remains an ad hoc assumption. Additionally, the linear potential is able (Christodoulou 1991) to stabilize disk galaxies without any need for dark matter, and the conformal gravity theory thus appears to be able to reproduce all the desirable aspects of galactic dark matter without needing the dark matter itself. Further, the conformal theory possesses one fewer free parameter per galaxy compared to dark halo models ( $M/L$  and  $\gamma$  instead of  $M/L$ ,  $\rho_0$  and  $r_0$ ). Consequently, according to the usual criteria for evaluating rival theories, as long as conformal gravity continues to hold up, it is to be preferred.

#### (4) Implications of Conformal Gravity for Clusters of Galaxies

In discussing the behavior of gravitating systems with a large number of degrees of freedom such as a cluster of galaxies, usually only average information such as a mean velocity dispersion is available rather than detailed features such as rotation curves which describe the motions of the individual constituents of the system. Consequently, the analysis is a strictly statistical one, and generally is based on the use of equations such as the collisionless Boltzmann equation and the assumption of virialization (see e.g. Binney and Tremaine 1987 for a comprehensive review). Since the standard discussion generally considers the two-body collision dependent term in the Boltzmann equation to be a local perturbation on the global mean field set up by the gravitational field of the rest of the particles in the cluster (treating each galaxy as a point particle for simplicity), we have to reexamine the entire formalism in light of the linear potentials of Eq. (3) which grow with distance and which can thus not be thought of as producing localizable collisions at all. Moreover, in a theory with rising potentials, we are not free to ignore the effects due to the particles in the rest of the Universe, so that even if a system such as a cluster of galaxies is geometrically isolated, that does not immediately mean that it is gravitationally isolated or that it is bound purely under its own self forces (something that of course would be the case in a strictly Newtonian theory where forces do fall with distance). Thus in order to fully apply our theory to clusters we not only need to develop an appropriate formalism for describing the local gravitational effects purely within a cluster, but in principle we also need to consider the coupling of the cluster to the rest of the Universe. To discuss the effect of such global coupling



requires developing a theory for the growth of inhomogeneities and galaxy formation, with the relevant issue for motions within clusters then being not so much the coupling of each cluster to the general Hubble flow produced by a homogeneous background distribution of sources, but rather its coupling to the deviations from that flow caused by the presence of inhomogeneities. Since a theory for the growth of inhomogeneities in conformal gravity has yet to be developed, we are currently unable to address this issue explicitly or explore its numerical consequences for cluster velocity dispersions (as a first step though we will determine below the region where the cluster density falls to the general cosmological background density and use this to ascertain where it is that the cluster actually ends). Consequently, for the purposes of this paper, we concentrate primarily on the local gravitational problem within a given cluster, and actually present a solution to this problem in the linear potential case based on the Bogoliubov, Born, Green, Kirkwood, and Yvon (BBGKY) hierarchy (this being a far more general statistical formalism than the collisionful Boltzmann equation to which it can actually reduce under specific assumptions (see e.g. Huang 1987 and Liboff 1990)). We first present some general features of the formalism, and then make an explicit application of it to the Coma cluster.

Before discussing this Liouville equation based statistical analysis, it is convenient to first examine some features which follow purely from the equations of motions of the cluster particles as they move in some general potential  $V(\bar{x})$  according to

$$\frac{d^2\bar{x}}{dt^2} = -\frac{\partial V(\bar{x})}{\partial \bar{x}} \quad (47)$$

(While we shall of course base our discussion on the non-relativistic geodesics of Eq. (47), we note in passing that while Eq. (47) correctly describes the coupling of a massive test particle to the Newtonian and linear potentials of Eq. (3), it does not incorporate any effect due to a direct interaction between the linear potentials of differing particles. In a more complete treatment such an effect would be generated via the two-body correction to one-body geodesic motion, an effect which for non-relativistic motions would be expected to only be a small perturbation on Eq. (47). Since the two-body problem in fourth order gravity is as far from solution as that in the standard second order theory, we shall simply ignore any such effects here.) Given Eq. (47), it then trivially follows that

$$\frac{1}{2} \frac{d^2}{dt^2} (\bar{x}^2) = v^2 - r \frac{\partial V(\bar{x})}{\partial r} \quad (48)$$

Thus for a cluster of  $N$  particles (with coordinates labeled by  $\alpha$  ( $\alpha = 1, \dots, N$ ), and equal mass  $m$  for simplicity), the trace of the moment of inertia tensor of the cluster obeys

$$\frac{1}{2} \frac{d^2 I}{dt^2} = \frac{1}{2} \frac{d^2}{dt^2} \left( m \sum_{\alpha=1}^N r_{\alpha}^2 \right) = m \sum_{\alpha=1}^N v_{\alpha}^2 - m \sum_{\alpha=1}^N r_{\alpha} \frac{\partial V(\bar{x}_{\alpha})}{\partial r_{\alpha}} \quad (49)$$

and thus directly yields for a spherically symmetric system with matter volume density  $\sigma(r)$  the familiar

$$\frac{1}{2} \frac{d^2 I}{dt^2} = 4\pi m \int_0^{\infty} dr r^2 \sigma(r) v^2(r) - 4\pi m \int_0^{\infty} dr r^2 \sigma(r) r V'(r) \quad (50)$$

Since Eq. (50) is based only on spherical symmetry with no commitment as to the explicit structure of the potential being needed, it thus immediately holds both in the Newtonian case and in our linear potential case as well, with the appropriate  $rV'(r)$  needed for Eq. (50) then being given directly by Eq. (25) which was explicitly derived earlier for the spherical case. Finally, should the entire cluster be in a steady state with  $\ddot{I} = 0$  (this is actually the key assumption as we will see below), we then obtain an expression for the

spatially averaged mean square virial velocity (here and throughout ‘av’ will mean averaged with respect to the spatial distribution)

$$N(v^2)_{av} = 4\pi \int_0^\infty dr r^2 \sigma(r) r V'(r) \quad (51)$$

in the spherically symmetric case, which can then readily be evaluated once an appropriate  $\sigma(r)$  is specified. (Even if  $\dot{I}$  is not in fact time independent, as long as  $\dot{I}$  remains bounded, Eq. (51) would still be valid provided it is reinterpreted as a long time average).

For our discussion of clusters below, we note that Eq. (51) admits of a local generalization. Since  $r^2$  is time independent in a circular orbit, these orbits actually satisfy Eq. (51) orbit by orbit and not merely in a statistical sense. Thus both the set of all circular orbits and the set of all non-circular ones must each satisfy Eq. (51) separately (the circular ones individually, but the non-circular ones only statistically). Thus suppose that some interior region (say from  $r = 0$  to some maximum  $r = r_m$  with  $N(r_m)$  particles) of the cluster has the property that all of its non-circular orbits actually stay within the region with little radial inflow or outflow between this region and the rest of the cluster. In this interior region then the set of all of the orbits (circular and non-circular combined) would then satisfy  $\dot{I}(r < r_m) = 0$  and yield the local

$$N(r_m)(v^2(r_m))_{av} = 4\pi \int_0^{r_m} dr r^2 \sigma(r) r V'(r) \quad (52)$$

for a cluster whose central region is in a steady state. For a Newtonian system Eq. (52) is completely self-contained since the total potential then only depends on the matter interior to the point of observation with Eq. (52) only receiving contributions from the matter interior to  $r_m$ . However, as can be seen from Eq. (25), in the linear potential case the total potential at a point is also sensitive to the matter exterior to that point, and thus the rest of the cluster exterior to  $r_m$  (and, in principle, the rest of the matter in the Universe as well) contributes non-trivially and potentially even significantly to the virial velocity within any virialized central region of radius  $r_m$ . As we will see below, the crucial issue in applying the virial to clusters will turn out to be determining just how big  $r_m$  might be, and in order to see how to address this point we turn to a statistical BBGKY analysis.

For a system of  $N$  particles moving under the conservative forces associated with equations of motion such as Eq. (47), the normalized (to one)  $6N$  dimensional distribution function  $f^{(N)}(\bar{w}_\alpha, t)$  ( $\bar{w}_\alpha = \{\bar{x}_\alpha, \bar{v}_\alpha\}$ ) obeys the Liouville equation

$$\frac{df^{(N)}}{dt} = \frac{\partial f^{(N)}}{\partial t} + \sum_{\alpha=1}^N \left[ \bar{v}_\alpha \cdot \frac{\partial f^{(N)}}{\partial \bar{x}_\alpha} - \frac{\partial V_\alpha}{\partial \bar{x}_\alpha} \cdot \frac{\partial f^{(N)}}{\partial \bar{v}_\alpha} \right] = 0 \quad (53)$$

where the potential  $V_\alpha$  seen by particle  $\alpha$  can be written as a sum of two-body potentials, viz.

$$V_\alpha = \sum_{\beta \neq \alpha} \phi_{\alpha\beta} \quad (54)$$

If the distribution function  $f^{(N)}(\bar{w}_\alpha, t)$  is both symmetric under exchange of any of the particles and sufficiently convergent asymptotically for all  $\bar{w}_\alpha$ , it then follows upon integrating Eq. (53) (see e.g. Binney and Tremaine 1987) that the one and two particle distribution functions

$$\begin{aligned} f^{(1)}(\bar{w}_1, t) &= \int f^{(N)} d^6 \bar{w}_2 \dots d^6 \bar{w}_N \\ f^{(2)}(\bar{w}_1, \bar{w}_2, t) &= \int f^{(N)} d^6 \bar{w}_3 \dots d^6 \bar{w}_N \end{aligned} \quad (55)$$

are related via

$$\frac{\partial f^{(1)}}{\partial t} + \bar{v}_1 \cdot \frac{\partial f^{(1)}}{\partial \bar{x}_1} = (N-1) \int \frac{\partial \phi_{12}}{\partial \bar{x}_1} \cdot \frac{\partial f^{(2)}}{\partial \bar{v}_1} d^6 \bar{w}_2 \quad (56)$$

In terms of the two-particle correlation function defined by

$$g(\bar{w}_1, \bar{w}_2, t) = f^{(2)}(\bar{w}_1, \bar{w}_2, t) - f^{(1)}(\bar{w}_1, t) f^{(1)}(\bar{w}_2, t) \quad (57)$$

and the conventional kinetic theory phase space density  $f(\bar{w}_1, t) = N f^{(1)}(\bar{w}_1, t)$  which is normalized according to

$$\int f(\bar{x}, \bar{v}, t) d^3 \bar{v} = \sigma(\bar{x}, t) \quad , \quad \int \sigma(\bar{x}, t) d^3 \bar{x} = N \quad (58)$$

we find that Eq. (56) then reduces (for large  $N$ ) to

$$\begin{aligned} & \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial t} + \bar{v} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{x}} - \frac{\partial V(\bar{x})}{\partial \bar{x}} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{v}} \\ &= N^2 \frac{\partial}{\partial \bar{v}} \cdot \int \frac{\partial \phi(\bar{x}, \bar{x}_2)}{\partial \bar{x}} g(\bar{x}, \bar{v}, \bar{x}_2, \bar{v}_2, t) d^3 \bar{x}_2 d^3 \bar{v}_2 \end{aligned} \quad (59)$$

where

$$V(\bar{x}) = \int \phi(\bar{x}, \bar{x}_2) f(\bar{x}_2, \bar{v}_2, t) d^3 \bar{x}_2 d^3 \bar{v}_2 = \int \phi(\bar{x}, \bar{x}_2) \sigma(\bar{x}_2, t) d^3 \bar{x}_2 \quad (60)$$

with Eq. (59) actually being exact in the large  $N$  limit. While the above derivation is completely standard, our point in repeating it here is to bring out the fact that there is no need to specify the explicit form of the potential in order to derive Eq. (59), with Eq. (59) thus still being valid even in the presence of our linear potential. We thus extend BBGKY (as opposed to the collisionful Boltzmann equation) to include linear potentials.

In order to extract out some general information from Eq. (59) we note first that if we set the two-body correlation  $g(\bar{w}_1, \bar{w}_2, t)$  to zero, Eq. (59) reduces to the Vlasov equation

$$\frac{\partial f(\bar{x}, \bar{v}, t)}{\partial t} + \bar{v} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{x}} - \frac{\partial V(\bar{x})}{\partial \bar{x}} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{v}} = 0 \quad (61)$$

which is often used in discussions of clusters, with  $V(\bar{x})$  of Eq. (60) then serving as the self-consistent mean gravitational field in which each particle moves. (Equation (61) is often also called the collisionless Boltzmann equation though in general there are in fact some differences between the Vlasov and collisionless Boltzmann equations which we comment on briefly below, but, independent of what name it may be given, the important point is that we are only able to use Eq. (61) for clusters when correlations are negligible). However, suppose we do not in fact drop the two-body correlation term in Eq. (59) but instead continue to carry it. Then, since its right hand side is a total divergence with respect to velocity, if we integrate Eq. (59) over  $d^3 \bar{v}$  the correlation term will simply make no contribution in this integration. (We assume here and throughout that the correlation term falls fast enough to compensate for the growing linear potential so that the surface terms are in fact negligible). Moreover, the velocity derivative term on the left hand side of Eq. (59) would similarly integrate away in this case, and on introducing the one-particle distribution averages (here and throughout ' $\langle \rangle$ ' will mean averaged with respect to the velocity distribution)

$$\begin{aligned} \sigma(\bar{x}, t) \langle v_i \rangle &= \int v_i f(\bar{x}, \bar{v}, t) d^3 \bar{v} \\ \sigma(\bar{x}, t) \langle v_i v_j \rangle &= \int v_i v_j f(\bar{x}, \bar{v}, t) d^3 \bar{v} \quad , \quad P_{ij} = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle \end{aligned} \quad (62)$$

( $i = 1, 2, 3$ ), we therefore obtain

$$\frac{\partial \sigma(\bar{x}, t)}{\partial t} + \frac{\partial}{\partial \bar{x}} \cdot (\sigma(\bar{x}, t) < \bar{v} >) = 0 \quad (63)$$

which we recognize as the standard kinetic theory continuity equation. Further, if we multiply Eq. (59) by  $v_i$  first and then integrate over all velocity, we next obtain

$$\begin{aligned} & \frac{\partial}{\partial t} (\sigma(\bar{x}, t) < v_i >) + \frac{\partial}{\partial x_j} (\sigma(\bar{x}, t) < v_i v_j >) + \sigma(\bar{x}, t) \frac{\partial V(\bar{x})}{\partial x_i} \\ &= \sigma(\bar{x}, t) \frac{\partial < v_i >}{\partial t} + \sigma(\bar{x}, t) < v_j > \frac{\partial < v_i >}{\partial x_j} + \frac{\partial}{\partial x_j} (\sigma(\bar{x}, t) P_{ij}) + \sigma(\bar{x}, t) \frac{\partial V(\bar{x})}{\partial x_i} \\ &= -N^2 \int \frac{\partial \phi(\bar{x}, \bar{x}_2)}{\partial x_i} g(\bar{x}, \bar{v}, \bar{x}_2, \bar{v}_2, t) d^3 \bar{v} d^3 \bar{x}_2 d^3 \bar{v}_2 \end{aligned} \quad (64)$$

following an integration by parts and the use of Eq. (63). Equation (64) thus differs from the familiar Jeans or Euler equation by virtue of the presence of the two-body correlation term on the right hand side. Finally, if we also contract with  $x_i$  and then integrate over all  $\bar{x}$  we obtain

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \int \sigma(\bar{x}, t) < \bar{x} \cdot \bar{v} > d^3 \bar{x} \right) - \int \sigma(\bar{x}, t) < v^2 > d^3 \bar{x} + \int \sigma(\bar{x}, t) < \bar{x} \cdot \frac{\partial V(\bar{x})}{\partial \bar{x}} > d^3 \bar{x} \\ &= -N^2 \int \bar{x} \cdot \frac{\partial \phi(\bar{x}, \bar{x}_2)}{\partial \bar{x}} g(\bar{x}, \bar{v}, \bar{x}_2, \bar{v}_2, t) d^3 \bar{x} d^3 \bar{v} d^3 \bar{x}_2 d^3 \bar{v}_2 \end{aligned} \quad (65)$$

which would recover the virial relation of Eq. (50) only if we were to set the correlation term to zero.

Since the virial relation of Eq. (50) is generally regarded as being an exact relation, it is necessary to explain why we are only able to recover it from Eq. (65) if there are no correlations. To this end, if we return to the exact BBGKY equation of Eq. (56) and proceed to average it just as we averaged Eq. (59), then the continuity equation of Eq. (63) would still obtain, but Eqs. (64) and (65) would respectively be replaced by

$$\begin{aligned} & \frac{\partial}{\partial t} (\sigma(\bar{x}, t) < v_i >) + \frac{\partial}{\partial x_j} (\sigma(\bar{x}, t) < v_i v_j >) \\ &= -N^2 \int \frac{\partial \phi(\bar{x}, \bar{x}_2)}{\partial x_i} f^{(2)}(\bar{x}, \bar{v}, \bar{x}_2, \bar{v}_2, t) d^3 \bar{v} d^3 \bar{x}_2 d^3 \bar{v}_2 \end{aligned} \quad (66)$$

and

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \int \sigma(\bar{x}, t) < \bar{x} \cdot \bar{v} > d^3 \bar{x} \right) - \int \sigma(\bar{x}, t) < v^2 > d^3 \bar{x} \\ &= -N^2 \int \bar{x} \cdot \frac{\partial \phi(\bar{x}, \bar{x}_2)}{\partial \bar{x}} f^{(2)}(\bar{x}, \bar{v}, \bar{x}_2, \bar{v}_2, t) d^3 \bar{x} d^3 \bar{v} d^3 \bar{x}_2 d^3 \bar{v}_2 \end{aligned} \quad (67)$$

which involve two-particle averages of the potential. Moreover, if the two-body potential has a power law dependence  $\phi_{12} = |\bar{x}_1 - \bar{x}_2|^n$ , we can then reexpress Eq. (67) as

$$\begin{aligned} & \frac{\partial}{\partial t} \left( \int \sigma(\bar{x}, t) < \bar{x} \cdot \bar{v} > d^3 \bar{x} \right) - \int \sigma(\bar{x}, t) < v^2 > d^3 \bar{x} \\ &= -\frac{nN^2}{2} \int \phi(\bar{x}, \bar{x}_2) f^{(2)}(\bar{x}, \bar{v}, \bar{x}_2, \bar{v}_2, t) d^3 \bar{x} d^3 \bar{v} d^3 \bar{x}_2 d^3 \bar{v}_2 \end{aligned} \quad (68)$$

a completely exact relation which should be regarded as the true virial in which the two-particle potential is averaged with the full two-body distribution thus allowing for the possibility that the particles can be

correlated. In fact only if particles are uncorrelated does Eq. (68) reduce to Eq. (50) with the correlation energy then dropping out. Usually in classical mechanics we define the potential energy of an  $N$  particle system to be that exhibited in Eq. (50). However, that energy is the work done in independently bringing each particle in from infinity one by one in the mean field provided by those that had already been brought in, and thus simply ignores any possible correlations, i.e. even though the insertion of the potential  $V_\alpha$  of Eq. (54) into Eq. (49) generates a two-body sum, by construction that sum is an uncorrelated one to thus yield the one-body Eq. (50). (In passing we note since the insertion of Eq. (54) into our starting point of Eq. (47) does not generate any such double sum, the equation of motion itself is always only a one-body equation independent of whether or not there are any two-body correlations - indeed the correlations of the BBGKY equations are generated statistically starting from the correlation insensitive Eq. (47). Now equations of motion such as Eq. (47) can also be derived starting from a Lagrangian (or a Hamiltonian) which involves the total potential energy written as the uncorrelated double sum  $\Sigma\phi(\alpha, \beta)/2$  over all  $\{\alpha, \beta \neq \alpha\}$ . However that double sum can only serve as the potential energy when there are no correlations, and even though its (uncorrelated) Euler-Lagrange variation does indeed lead to Eq. (47), Eq. (47) is also valid even in the presence of correlations when it must instead be associated with Eq. (68), to thereby account in general for both equilibrium and non-equilibrium situations. In classical kinetic theory then the equation of motion and the Liouville operator of Eq. (53) thus have primacy over the Lagrangian which is only readily definable for uncorrelated (equilibrium) systems.) Thus in the presence of correlations it is Eq. (68) rather than Eq. (50) that should be used in general, with Eq. (50) only emerging at times late enough that all correlations have had time to die out, a steady state situation in which the system is conventionally referred to as being virialized. Hence we see that in the presence of correlations we must not merely not use Eq. (50), but we also should not expect  $\ddot{I}$  to be zero. Moreover, since a steady state solution to the correlationless Vlasov equation would only mildly constrain the one-particle distribution function by requiring it to be a function only of the total single particle energy  $mv^2/2 + mV(\bar{x})$  (and also the angular momentum if the system is not isotropic), we see that the specific dependence on energy which eventually will emerge at late times for any given system must be fixed by the manner in which the correlations approach zero at late times, something which can only be fixed by the entire BBGKY all order hierarchy, with the next equation in the hierarchy for instance being

$$\begin{aligned} & \frac{\partial f^{(2)}}{\partial t} + \bar{v}_1 \cdot \frac{\partial f^{(2)}}{\partial \bar{x}_1} + \bar{v}_2 \cdot \frac{\partial f^{(2)}}{\partial \bar{x}_2} - \frac{\partial \phi_{12}}{\partial \bar{x}_1} \cdot \left( \frac{\partial f^{(2)}}{\partial \bar{v}_1} - \frac{\partial f^{(2)}}{\partial \bar{v}_2} \right) \\ &= (N-2) \frac{\partial}{\partial \bar{v}_1} \cdot \int \frac{\partial \phi_{13}}{\partial \bar{x}_1} f^{(3)} d^6 \bar{w}_3 + (N-2) \frac{\partial}{\partial \bar{v}_2} \cdot \int \frac{\partial \phi_{23}}{\partial \bar{x}_2} f^{(3)} d^6 \bar{w}_3 \end{aligned} \quad (69)$$

in an essentially intractable infinite chain. Thus we see that the two-body correlations have to play a crucial role in the evolution of the system which imprints itself on the ultimate late time equilibrium distribution function (assuming that equilibrium ever occurs) in an essentially completely unknown way for which there is currently no clear guidance.

Before leaving the general discussion of the BBGKY hierarchy, it is of some interest to compare the BBGKY approach with that of the standard Boltzmann equation approach. In kinetic theory the great practical difficulty in using the BBGKY hierarchy is that knowledge of the form of the two-body distribution is needed in order to determine the one-body distribution  $f(\bar{x}, \bar{v}, t)$ . As an alternative to this chain we could instead consider the master equation which explicitly counts how many particles enter and leave a given region of phase space through two-body collisions. Thus in general if particles are sitting in some global external potential  $V_{ext}(\bar{x})$  (i.e. truly external to the system of particles of interest) and undergo local scattering in and out of some region of phase space  $\bar{w}$  through two-body interparticle collisions, then in

general we may write (e.g. Binney and Tremaine 1987)

$$\begin{aligned} & \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial t} + \bar{v} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{x}} - \frac{\partial V_{ext}(\bar{x})}{\partial \bar{x}} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{v}} \\ &= \int [\Psi(\bar{w} - \Delta\bar{w}, \Delta\bar{w})f(\bar{w} - \Delta\bar{w}) - \Psi(\bar{w}, \Delta\bar{w})f(\bar{w})]d^6\Delta\bar{w} \end{aligned} \quad (70)$$

where  $\Psi(\bar{w}, \Delta\bar{w})d^6\Delta\bar{w}$  is the probability per unit time that particles with coordinates  $\bar{w}$  will scatter into phase space volume  $d^6\Delta\bar{w}$ . As derived, Eq. (70) requires the explicit splitting of the potential of Eq. (47) into clearly distinguishable external and scattering pieces (which typically in standard kinetic theory would mean some external electromagnetic or gravitational field applied to a gas which experiences local molecular interactions). While exact, Eq. (70) is just as intractable as Eq. (59) since the two-body distribution is again involved this time through the presence of the two-body scattering probability  $\Psi(\bar{w}, \Delta\bar{w})$ . In order to proceed, some approximation needs to be made in which the two-body distribution can be expressed in terms of  $f(\bar{x}, \bar{v}, t)$  in some way. Boltzmann's own approach was to begin with the master equation, restrict to local (molecular) two-body scattering and proceed from the assumption of molecular chaos. Alternative approaches which work down from the BBGKY equations and lead to similar results in the short-range molecular case are described in Huang (1987) and Liboff (1990), with all of these approaches leading to the collisionful Boltzmann equation for the one-particle distribution function, viz.

$$\begin{aligned} & \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial t} + \bar{v} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{x}} - \frac{\partial V_{ext}(\bar{x})}{\partial \bar{x}} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{v}} \\ &= \int |\bar{v} - \bar{v}_2| \sigma(\Omega) (f(\bar{x}, \bar{v}', t)f(\bar{x}, \bar{v}_2', t) - f(\bar{x}, \bar{v}, t)f(\bar{x}, \bar{v}_2, t)) d^3\bar{v}_2 d\Omega \end{aligned} \quad (71)$$

written here quite generally for a system of particles in an external potential  $V_{ext}(\bar{x})$  undergoing local momentum conserving two-body collisions  $\bar{v} + \bar{v}_2 \rightarrow \bar{v}' + \bar{v}_2'$  through scattering angle  $\Omega$  with differential cross section  $\sigma(\Omega)$ .

For a purely self-gravitating system such as a cluster there is no explicit external  $V_{ext}(\bar{x})$  term to begin with (since the right hand side of the master equation would then necessarily (by definition) have to contain the effects of all of the gravitational scatterings within the self-gravitating cluster). Thus the first key question to ask in possible applications of Eq. (71) to clusters is whether Eq. (71) (with  $V_{ext}(\bar{x}) = 0$ ) actually follows from the BBGKY hierarchy or master equation at all. (If it were to do so, then the collisionless Boltzmann equation would technically then simply be  $\partial f / \partial t + \bar{v} \cdot \partial f / \partial \bar{x} = 0$  rather than the Vlasov equation of Eq. (61).) Apart from the issue of the a priori validity of Eq. (71) in the self-gravitating case, it is further generally assumed in the literature that in the self-gravitating case the total gravitational potential of Eq. (47) may actually be divided up into separate local and global contributions. Even though it is not immediately clear how this can be done in general (and certainly not clear in cases with long range potentials which grow with distance), and despite the fact that it might even involve a double counting problem,  $V_{ext}(\bar{x})$  is usually identified with the mean gravitational field  $V(\bar{x})$  of Eq. (60) and Eq. (71) is replaced by

$$\begin{aligned} & \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial t} + \bar{v} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{x}} - \frac{\partial V(\bar{x})}{\partial \bar{x}} \cdot \frac{\partial f(\bar{x}, \bar{v}, t)}{\partial \bar{v}} \\ &= \int |\bar{v} - \bar{v}_2| \sigma(\Omega) (f(\bar{x}, \bar{v}', t)f(\bar{x}, \bar{v}_2', t) - f(\bar{x}, \bar{v}, t)f(\bar{x}, \bar{v}_2, t)) d^3\bar{v}_2 d\Omega \end{aligned} \quad (72)$$

to yield an equation which then of course would yield the Vlasov equation where we to then drop the collision integral term. While Eq. (72) even appears plausible in that it identifies  $V_{ext}(\bar{x})$  with the mean gravitational

field  $V(\bar{x})$ , there would not appear to be any explicit formal derivation of Eq. (72) for self-gravitating systems in the literature starting from either Eq. (59) or Eq. (70), even apparently in the well-studied  $1/r$  potential case. Thus the second key issue for clusters then is not so much what the implications of Eq. (72) might be, but rather how valid it in fact is in the first place. However, in passing, it is interesting to note (see e.g. Huang 1987) that since elastic two-body collisions conserve the total momentum, the total energy, and the total number of particles, the integration over all velocities of the product of any of these conserved quantities with the collision integral will automatically give zero. The collision integral term thus makes no contribution in these averagings, with averaging of Eq. (72) thus immediately yielding (for any potential in fact for which Eq. (72) is valid) the correlationless version of Eqs. (63) - (65), i.e. the form these equations would take if the two-body correlation function  $g(\bar{w}_1, \bar{w}_2, t)$  were set to zero in them. Thus, once we have the collisionful Boltzmann equation in the form of Eq. (72) at all, the standard gravitational Jeans and virial equations would follow, even in the presence of the collision integral term, without any need to invoke the Vlasov equation. The Jeans equation is thus in principle valid even in the presence of the two-body collisions of Eq. (72), with the Jeans equation then holding even for distributions which do not obey the Vlasov equation, so that the Vlasov equation would then only be sufficient to yield the Jeans equation, but not necessary.

To explore the issue of whether Eq. (72) is in fact actually valid in the first place, let us suppose for the sake of the argument that Eq. (72) is in fact valid for Newtonian clusters. For them the collision integral term turns out to be negligible on small time scales. Specifically, since the two-body cross section in a  $1/r$  potential is of order  $\sigma \sim m^2 G^2 / v^4$  while a typical velocity is of order  $v^2 \sim NmG/R$  for an  $N$  particle cluster with radius  $R$ , the collision integral term in Eq. (72) is of order  $f/Nt_c$  where  $t_c = R/v$  is the cluster crossing time. Thus only after  $N$  crossing times will the collision integral compete with the left hand side of Eq. (72), with Eq. (72) thus reducing to the Vlasov equation of Eq. (61) at short times, a condition which only mildly constrains the distribution function by requiring it to be a function only of the total single particle energy  $mv^2/2 + mV(\bar{x})$  (should the distribution function be time independent that is in this epoch, with this assumption actually being an additional independent constraint beyond simply assuming the actual validity of the Vlasov equation itself). On the other hand, at very late times, through the very fact of rescattering, Eq. (72) would lead us to an actual specification of the functional dependence of  $f(\bar{x}, \bar{v}, t)$  on the energy, i.e. the Maxwell-Boltzmann distribution  $f \sim \exp(-(mv^2/2 + mV(\bar{x}))/kT)$  which is an exact solution to the collisionful Eq. (72) in which the collision integral term vanishes non-trivially through the vanishing of  $f(\bar{v}')f(\bar{v}_2') - f(\bar{v})f(\bar{v}_2)$ . Thus, unlike the collisionful Boltzmann equation, a collisionless Boltzmann equation simply does not contain enough information to fix the distribution function completely. Hence if the collision integral is indeed numerically small in some kinematic regime, the distribution function which eventually results in that regime must be fixed by something else, with the only apparent other candidate being the multi-body correlations of the entire all order BBGKY hierarchy, in which case Eq. (72) could not have been valid at that time in the first place. Or stated differently, if the Vlasov equation is valid at early rather than late times, then nothing is available to fix the form of the distribution function at that earlier time. Thus the validity of the collisionful Eq. (72) for Newtonian clusters would require the following somewhat peculiar time development profile. First, at some very early time even prior to the onset of Eq. (72) the two-body correlation term in Eq. (59) would have to be important. Then it would have to gradually dampen to zero as it forces the system into some particular solution to the Vlasov equation of Eq. (61) with some particular dependence on the energy then being determined (if  $\partial f(\bar{x}, \bar{v}, t)/\partial t$  is in fact zero in this regime). Then at later times still the collision integral term effects would have to become important (even though the two-body correlation  $g(\bar{w}_1, \bar{w}_2, t)$  would still be required to be negligible because of the continuing imposition of molecular chaos) and move the system away from being in a solution to the Vlasov equation, and then

finally at the very latest times the system would then have to thermalize into Maxwell-Boltzmann. There might thus appear to be some difficulties for clusters if the collisionful Boltzmann equation is ever valid in the self-gravitating case, with the smallness of the right hand side of Eq. (72) apparently turning out to be something of a minus rather than a plus for Newtonian clusters. On the other hand, there would not appear to be any formal difficulty in either the Newtonian or linear potential case in having the cluster evolve directly via BBGKY into a late time Vlasov equation through the late time vanishing of the correlation functions in a time development which imprints itself on the resulting distribution function while never passing through any Boltzmann equation regime at all. Thus we shall restrict our discussion of clusters to the use of the BBGKY hierarchy without regard to the Boltzmann equation at all. (Of course, in practice if one only uses the Vlasov equation of Eq. (61), it does not particularly matter whether it came from BBGKY or from the collisionful Boltzmann equation anyway. It would only matter if one wants to follow the approach to equilibrium of a not yet fully virialized system). While there may not be any formal difficulty in using our preferred choice of the BBGKY hierarchy, there is of course a great practical difficulty though, since without knowledge of the time development of the correlation functions, it is not possible to ascertain into which particular distribution function the system eventually does develop in any such late time Vlasov equation regime. Because of this, we shall below only seek implications of the BBGKY approach which require no knowledge of the explicit form of the one-particle distribution function (a procedure which incidentally releases us from needing to invoke any of the popular model distribution functions often considered in cluster studies, models for which the literature seems not to present any a priori justification).

While a first principles evaluation of the validity of the virial must await a determination of the two-body correlation function, it is possible to establish some phenomenological expectations using observed cluster data. For the most studied cluster, the Coma cluster, the relevant data may be found in Kent and Gunn (1982), The and White (1986), and White et al (1993). At the distance of Coma an arc minute is  $20/h$  kpc (for a Hubble parameter  $H_0 = 100h$  km/sec/Mpc), so that the standard Abell radius is  $75'$ . The surface brightness may be approximately fitted (see below) by a modified Hubble profile ( $\sim 1/(R^2 + R_0^2)$ ) with a core radius  $R_0 = 9.23'$ , and the observed cluster data go out about to  $3^\circ \simeq 20R_0$  or so. (It is not immediately clear where the cluster actually ends, a point we examine below.) White et al (1993) quote a total blue surface luminosity within the Abell radius of  $L_B = 1.95 \times 10^{12}/h^2 L_{B\odot}$ , and a mean projected line of sight velocity of 970 km/sec for a convenient magnitude limited cut on the data which restricts to  $R \leq 120'$  (the revised binning of The and White to the original velocity data of Kent and Gunn is reproduced here as Fig. (3)). For such a mean velocity, the time required to cross the associated  $240'$  diameter is  $1.5 \times 10^{17}/h$  sec, which is of order  $1/2H_0$ , i.e. of order half a Hubble time, and thus we should not expect the entire cluster to have yet had time to virialize. Hence, for the purposes of this study, we shall simply assume that only the inner region cluster core - a region which still turns out to contain a sizable fraction of the entire cluster - has so far virialized. (While we can readily assert that we would not expect virialization after only one crossing time, we note in passing that having a time long enough for quite a few crossing times is not in and of itself sufficient to ensure that we then would have virialization, since the relevant time scale is not the crossing time but the BBGKY correlation relaxation time, a time which is currently unknown; and in general it would appear to require a somewhat subjective judgment to say exactly just how big a fraction of the entire cluster has yet had time to virialize.) Further, the very fact that the cluster surface brightness does fall so slowly - unlike the rapid, exponential drop within the much smaller individual spiral galaxies - could also be an indicator that the entire cluster has not in fact yet had enough time to have virialized completely by compactifying itself into a relatively small volume with a much more steeply declining surface brightness (or to have yet succeeded in decoupling itself from the background provided by the rest of the Universe either for that matter). Moreover, while the potentials in the conformal theory grow with distance, tidal forces will



still fall (linearly) with distance and be proportional to the ratio of the size of orbit being perturbed to the distance between the two systems of interest. Thus while the rotation curves of individual spiral galaxies are essentially unaffected by the presence of any nearby galaxy, clusters in any nearby supercluster would, given their huge amount of matter, be expected to have some influence on the larger orbits of a given cluster of interest while only marginally affecting the smaller orbits, so again virialization is more reasonable for the core than for the periphery.

Before we proceed to get an actual numerical estimate for the virial velocity in the conformal gravity theory, we note first that since, as can be seen from Fig. (3), the projected line of sight velocity falls with distance, this might immediately appear to exclude any theory in which the potential grows rather than falls with distance. However, it turns out that the projected velocity can still actually fall in our theory simply because of the difference between the full three-dimensional and the projected two-dimensional averaging procedures. For a spherically symmetric system the projected line of sight distribution average  $\langle \sigma_p^2(R) \rangle$ , where  $R$  is the impact parameter, is related to the previously introduced three-dimensional radial and tangential velocity distribution averages according to

$$I(R) \langle \sigma_p^2(R) \rangle = 2 \int_R^\infty \frac{dr \sigma(r)}{r(r^2 - R^2)^{1/2}} (\langle v_r^2 \rangle (r^2 - R^2) + \langle v_\theta^2 \rangle R^2) \quad (73)$$

(The surface mass density  $I(R)$  is related to the volume mass density  $\sigma(r)$  via Eq. (26).) For a system which has already virialized, we may drop both the correlation term and the explicit average velocity time derivative term from Eq. (64), with a spherically symmetric steady state Jeans equation then yielding

$$\frac{d}{dr}(\sigma(r) \langle v_r^2 \rangle) + \frac{2\sigma(r)}{r}(\langle v_r^2 \rangle - \langle v_\theta^2 \rangle) = -\sigma(r)V'(r) \quad (74)$$

While it is not possible to solve Eqs. (73) and (74) in a closed form without further input, it is possible to extract out some general features from a study of some simple cases. If for instance we assume that the system is isotropic (so that  $\langle v_r^2 \rangle = \langle v_\theta^2 \rangle$ ), Eq. (73) then yields a closed form for the dependence of  $\langle \sigma_p^2(R) \rangle$  on the potential, viz.

$$I(R) \langle \sigma_p^2(R) \rangle = 2 \int_R^\infty dr \sigma(r) (r^2 - R^2)^{1/2} V'(r) \quad (75)$$

Similarly, if we assume that the system is purely circular ( $\langle v_r^2 \rangle = 0$ ), we obtain

$$I(R) \langle \sigma_p^2(R) \rangle = R^2 \int_R^\infty dr \frac{\sigma(r) V'(r)}{(r^2 - R^2)^{1/2}} \quad (76)$$

while a purely radial system ( $\langle v_\theta^2 \rangle = 0$ ) yields

$$I(R) \langle \sigma_p^2(R) \rangle = \frac{1}{R} \int_R^\infty dr \sigma(r) V'(r) \left( r^2 \arcsin \left( \left( 1 - \frac{R^2}{r^2} \right)^{1/2} \right) - R(r^2 - R^2)^{1/2} \right) \quad (77)$$

Thus the radial case will tend to emphasize small  $R$ , the circular case large  $R$ , and the isotropic case intermediate  $R$ .

To see how this works out in practice we can evaluate Eqs. (75) - (77) for the illustrative case of a cluster with a modified Hubble surface profile. Since such a slowly falling surface density would yield an infinite number of particles if integrated to infinity, we must cut off the density at some maximum radius  $R_M$ . However, the very act of cutting off the surface brightness has some explicit consequences for the Abel transform of Eq. (26). If, for instance, we simply give the surface brightness the sharp cut-off  $I(R)\theta(R - R_M)$ ,

its insertion into the Abel transform would then induce an unphysical singular term  $-I(R_M)/\pi(R_M^2 - r^2)^{1/2}$  in the volume density coming from the delta function derivative of the theta function. (This singularity is needed to recover  $I(R)$  when integrating back over the volume density since the range  $R$  to  $R_M$  for the volume density integration shrinks to zero as  $R$  approaches  $R_M$ .) However, there is no such problem in giving the volume density a sharp cut-off instead, since the generic matter volume density

$$\sigma(r) = \frac{\sigma_0}{(r^2 + R_0^2)^{3/2}} \theta(R_M - r) \quad (78)$$

then yields

$$I(R) = \frac{2\sigma_0(R_M^2 - R^2)^{1/2}}{(R^2 + R_0^2)(R_M^2 + R_0^2)^{1/2}} \theta(R_M - R) \quad (79)$$

as the generic matter surface density, a surface density which now nicely vanishes smoothly rather than sharply at the cut-off, and consequently in our explicit numerical fitting to the Coma cluster to be presented below we shall actually use Eq. (79) as a model for the surface brightness rather than just the plain  $2\sigma_0/(R^2 + R_0^2)$ . Given the generic Eqs. (78) and (79) it is now straightforward to evaluate the various line of sight velocities for both the Newtonian and the linear potential cases via the direct use of Eq. (25), and we present the resulting velocity curves (calculated with  $R_M = 20R_0$  for explicitness) as Figs. (4) and (5) respectively. As can be seen, in the Newtonian case all the three discussed possibilities have their maxima at small radii (the purely circular case is actually asymptotically flat in projection), while the linear potential case shows a radically different behavior. Specifically, the pure radial case peaks at very small  $R$ , the pure isotropic case at  $R_M/2$ , and the pure circular case at  $R_M$ . Thus, depending on the radial to tangential velocity ratio, in fully virialized clusters it is quite possible for the projected line of sight velocity to fall at large distances even in a theory with rising potentials. (Essentially at large impact parameters the amount of cluster material along a line of sight goes to zero faster than the rate at which the potential grows). We thus identify a somewhat unusual projection effect, and see that in general the curves of Figs. (4) and (5) could eventually turn out to be very useful in discriminating between Newtonian and linear potentials in systems which are in fact fully virialized.

Turning now to clusters which have not yet had time to virialize completely, we first evaluate the fractional amount of matter contained in the core region. For the densities of Eqs. (78) and (79) we can readily evaluate the fractional amount of matter  $4\pi \int dr' r'^2 \sigma(r')/N$  within a given volume of radius  $r$  and the fractional amount of matter  $2\pi \int dR' R' I(R')/N$  within a given projected surface of impact parameter  $R$ . As we can see from the curves of Fig. (6) (which are calculated with  $R_M = 20R_0$  for explicitness), one quarter of the matter by volume is contained within  $r < 2.5R_0$  and one half within  $r < 5R_0$ , while one half of the matter by surface is contained within  $R < 4R_0$ . The central region of the cluster thus contains a sizable portion of the entire amount of matter in the cluster, so that a virialization of only the inner region of the cluster is not insignificant. From the data we have only the projected two-dimensional line of sight velocity  $< \sigma_p^2(R) >$  which is inconveniently related in Eq. (73) to an integration of the three-dimensional radial and tangential velocities (the ones we actually calculate by the virial) over both the inner and outer regions of the cluster. However, an integration of Eq. (73) itself over a sphere of radius  $r_m$  then crucially projects out the undesired unvirialized  $r > r_m$  region from the  $R \leq r \leq R_M$  integration range involved in Eq. (73) to yield

$$\begin{aligned} & 2\pi \int_0^{r_m} dR R I(R) < \sigma_p^2(R) > \\ &= 2\pi \int_0^{r_m} dr r^2 \int_0^\pi d\theta \sin\theta \sigma(r) (< v_r^2 > \cos^2\theta + < v_\theta^2 > \sin^2\theta) \end{aligned}$$

$$= \frac{4\pi}{3} \int_0^{r_m} dr r^2 \sigma(r) (< v_r^2 > + 2 < v_\theta^2 >) \quad (80)$$

a relation which is actually completely general and which involves no assumptions regarding the relative strengths of the mean square radial and tangential velocities. From Eq. (80) we see that in general the averaged squared line of sight velocity is thus one third of the averaged squared three dimensional velocity. To relate the right hand side of Eq. (80) to the potential we appeal to Eq. (64) and note that if the two-body correlation term vanishes for  $r < r_m$  (and if  $\partial < v_i > / \partial t = 0$  of course), then the steady state Jeans equation of Eq. (74) will hold for  $r < r_m$ . Multiplying Eq. (74) by  $r^3$  and then integrating enables to express the spatial average  $(\sigma_p^2(r_m))_{av}$  for the region  $r < r_m$  purely in terms of virialized region quantities alone according to our final, key relation

$$2\pi(\sigma_p^2(r_m))_{av} \int_0^{r_m} dR R I(R) = \frac{4\pi}{3} \int_0^{r_m} dr r^2 \sigma(r) r V'(r) \quad (81)$$

(The integration of  $r^3$  times Eq. (74) involves surface terms at  $r = 0$  and  $r = r_m$ . The one at zero vanishes kinematically (since  $< v_r^2 >$  vanishes no faster than  $1/r^2$  according to Eq. (74)), while the other one leads to a dependence on  $< v_r^2(r_m) >$ . Our assumption that the  $r < r_m$  region is virialized requires the vanishing of this term in precisely the manner which led us to the partial virial of Eq. (52) which we discussed above.). For our illustrative volume density of Eq. (78) (with typical cut-off of  $R_M = 20R_0$  for the matter distribution) the local virials associated with Eq. (81) are readily evaluated, with the partial virial spatially averaged root mean square projected velocity  $\sigma_p(r_m) = ((\sigma_p^2(r_m))_{av})^{1/2}$  being plotted as a function of  $r_m$  in Fig. (7) for both the Newtonian and linear cases. In Fig. (7) the Newtonian virial is normalized to  $(N\beta_{gal}c^2/R_0)^{1/2}$  while the linear virial is normalized to  $(N\gamma_{gal}c^2R_0)^{1/2}$  where  $\beta_{gal}$  and  $\gamma_{gal}$  are individual galactic potential parameters and  $N$  is the total number of galaxies contained in the entire cluster - and not just the number  $N(r_m)$  contained in  $r \leq r_m$ . Since the Newtonian potential contribution of Eq. (25) is only sensitive to the matter interior to  $r_m$ , the Newtonian local virial very quickly levels off, but since the linear potential also feels the matter exterior to  $r_m$ , its associated local virial velocity continues to rise all the way to  $r = R_M$ . Thus the linear potential case is far more sensitive to how much of the cluster is virialized than the Newtonian one. (Thus in passing we note that since the Newtonian virial is so insensitive to how much of the cluster has in fact virialized, an application of Eq. (81) to the core will give predictions which are extremely close to those made under the assumption that the entire cluster is virialized, with the use of Eq. (81) in the standard Newtonian theory thus essentially being immune to the issue of how much of the cluster has in fact virialized.)

In order to apply the virial of Eq. (81) to the Coma cluster, we need first to study the implications of using the cut-off  $I(R)$  of Eq. (79) as a model for the surface brightness of Coma. Rather than fit this  $I(R)$  to the surface brightness, we instead opted to fit  $RI(R)$  (this being the quantity which actually appears in the virial equations) to  $R$  times the surface brightness so as to ensure the correct overall normalization. (The core virial velocities we obtain below turn out to be insensitive to this prescription). We can then fit the Coma surface brightness data with  $R_0 = 9.23'$ ,  $R_M = 20R_0 = 185'$ , and normalization  $\sigma_0/R_0^3 = 0.016$  galaxies per cubic arc min. Giving each galaxy an average blue luminosity of  $5.99 \times 10^9/h^2 L_{B\odot}$ , then yields the requisite total  $1.95 \times 10^{12}/h^2 L_{B\odot}$  surface blue luminosity within the Abell radius, to thus fully specify  $I(R)$ . Using as typical the mass to light ratio  $M/L_B = 5.6hM_\odot/L_{B\odot}$  which we obtained for the galaxy NGC 3198 (we adjust here for the fact that the fits of Fig. (1) were based on data which were obtained using an adopted value of  $h = 0.75$  for each of the three regular spirals in our sample) enables us to determine the mass volume density associated with  $\sigma(r)$ . It is very convenient to express this mass density in units of the standard critical density  $\rho_c = 3H_0^2/8\pi G$ , and we find that  $\sigma(0') = 241.5\rho_c$ ,  $\sigma(56.8') = \rho_c$ ,  $\sigma(120') = 0.11\rho_c$ , and  $\sigma(185') = 0.03\rho_c$ . The cluster is thus apparently merging with the general cosmological background at

no more than  $185'$  or so, and would be restricted to the first  $57'$  (a region which contains 57% of the total matter in the cluster by volume according to Fig. (6)) if the density of the Universe is critical. Thus in a low density Universe we would put the edge of the cluster at  $185'$ , while in a high density one we would only consider the potentials of the first  $57'$  of data as contributing to the velocity dispersion, with the next  $128'$  of data then only contributing along with the rest of the galaxies in the Universe to the general Hubble flow. (Noting that the conventional estimation of the cosmological ratio  $\rho/\rho_c$  is made in comoving coordinates while our analysis here involves the same ratio in static coordinates, our determination of where the static cluster actually merges with the comoving background is thus perforce only a rough estimate.) Since the actual density of the Universe represents one of the key unknown issues in cosmology, we shall calculate core virial velocities for both the high and low density Universe cases, and actually find below that the values that we then obtain turn out to be insensitive to where the cluster ends. (In a recent paper (Mannheim 1992) it was shown that the relativistic cosmology associated with conformal gravity possesses no flatness problem. Unlike the standard Einstein theory the conformal theory thus needs no inflationary era, and its cosmological matter density is not required to obey  $\rho = \rho_c$ . Given the fact that conformal gravity also appears to be able to eliminate the need for galactic scale dark matter, it can thus naturally accommodate a  $\rho < \rho_c$  Universe. Nonetheless, for phenomenological completeness we shall study the conformal theory virial velocity predictions for both high and low density Universes, with the core velocities turning out to be insensitive to this whole issue anyway.)

We proceed now to an actual evaluation of the virial velocities in our model. Our above discussion of the mass density of Coma fixes the overall normalization of the Newtonian potential contribution to the virial, while taking as typical the NGC 3198 gamma to light ratio of  $9.2 \times 10^{-40} h^3 / \text{cm} / L_{B\odot}$  obtained earlier then enables us to fix the overall normalization of the linear potential contribution as well. Thus for a Coma cluster composed solely of luminous matter alone, the overall normalizations  $(N\gamma_{gal}c^2R_0)^{1/2}$  and  $(N\beta_{gal}c^2/R_0)^{1/2}$  needed for Fig. (7) take respective values of 10960 km/sec and 576 km/sec for a cluster cut off at  $R_M = 20R_0$  ( $N = 425$  galaxies). From Fig. (7) we thus see that in the absence of any dark matter the luminous Newtonian contribution to the virial is negligibly small, while, on the other hand, the linear contribution associated with the luminous matter is substantial. Specifically, if the entire  $R_M = 20R_0$  cluster is virialized Eq. (81) yields a virial velocity  $\sigma_p(20R_0) = 10178$  km/sec, while also yielding partial virial velocities  $\sigma_p(R_0) = 1089$  km/sec,  $\sigma_p(1.5R_0) = 1678$  km/sec,  $\sigma_p(2R_0) = 2195$  km/sec, and  $\sigma_p(6.15R_0) = 5018$  km/sec in various inner regions. Similarly, if we cut off the cluster at  $56.8' = 6.15R_0$  (to yield  $N = 242$  galaxies,  $(N\gamma_{gal}c^2R_0)^{1/2} = 8261$  km/sec,  $(N\beta_{gal}c^2/R_0)^{1/2} = 435$  km/sec) we obtain the partial virial velocities  $\sigma_p(R_0) = 1028$  km/sec,  $\sigma_p(1.5R_0) = 1583$  km/sec,  $\sigma_p(2R_0) = 2070$  km/sec, and  $\sigma_p(6.15R_0) = 4885$  km/sec. The core region velocities are thus essentially insensitive to whether we use a high or low density Universe cut-off. (This may be understood directly from the potential of Eq. (25), since while that potential is sensitive to points exterior to the point of observation, their contribution is proportional to  $r^2$  which is small in the inner core region, to thus prevent the region outside of the core from making any substantial contribution to core region virial velocities). From the data points of Fig. (3) we find that the numerical average of the first four bins of data ( $R \leq 1.3R_0$ ) is  $1200 \pm 195$  km/sec, while that of the first five bins ( $R \leq 1.7R_0$ ) is  $1185 \pm 195$  km/sec. Before we assess the significance of these numbers, it is important to note that once less than the entire spherical cluster is virialized, then any given line of sight through the sphere, even those at small impact parameter  $R$ , will pass through both virialized and non-virialized regions (since the integral in Eq. (73) is from  $R$  all the way to the cluster cut-off  $R_M$ , and not merely to the virialization cut-off  $r_m$ ), so that the detected projected velocity at that  $R$  will include some non-virialized contributions as well. For instance, if  $r \leq 2.5R_0$  is virialized, then out of a  $20R_0$  cluster the percentage of line of sight material which involves unvirialized radii  $r > 2.5R_0$  is 25% at  $R = 1.5R_0$ , 44% at

$R = 2R_0$ , and of course 100% at  $R = 2.5R_0$ . Thus the very use of Fig. (3) to estimate a magnitude for a virialized  $\sigma_p(r_m)$  becomes suspect once the cluster is less than fully virialized. (In passing we note that with a value of  $(N\gamma_{gal}c^2R_0)^{1/2} = 10960$  km/sec, the projected curves of Fig. (5) would far overshoot the data for a fully virialized Coma cluster. However, even for just an inner region virialization, we still could not try to fit these curves for small  $R$  since even small  $R$  requires a knowledge of the large  $r$  behavior (up to  $R_M$ ) of the cluster in the integrals of Eqs. (75-77)). Fortunately, however, Eq. (81) only involves integrating up to  $r_m$ , and since it requires no knowledge of the distribution function or of the radial to tangential velocity admixture either, it would appear to be far the most reliable quantity to study, especially in only partially virialized systems. Conformal gravity would thus appear to have no difficulty accommodating a virialized inner cluster region of the order of  $r_m \sim 1.5R_0$  without needing to invoke dark matter, and given the just noted limitation on the use of the data of Fig. (3) in partially virialized systems, the theory could possibly even accommodate up to  $r_m \sim 2.5R_0$ , a region which contains close to one quarter by volume of all of the matter in the entire  $185'$  of the cluster. Moreover, given the relevant time scales which were discussed above, it would even appear to be quite reasonable to expect inner region virialization up to one or two scale lengths or so. While we would certainly not expect any larger a portion of the cluster to have yet virialized, a first principles determination of the two-body correlation function and of its potential impact on Eqs. (74), (75-77), and (81) could nonetheless prove to be very instructive, and might possibly even turn out to be definitive for the theory. (It is also possible to test the conformal theory in a way which is actually insensitive to how big a fraction of the cluster has in fact virialized, viz. cluster gravitational lensing which responds to all the matter in the cluster virialized or not; thus a yet to be made study of the conformal theory predictions for lensing should eventually provide an independent and definitive way of testing the theory on whole cluster scales.) Other than this issue though, it would appear that, in the first instance at least, the conformal gravity theory is indeed capable of meeting the demands of cluster virial velocity data, with the linear potential theory thus readily being extendable from galactic scales up to the much larger ones associated with clusters of galaxies without encountering any major difficulty.

## (5) Implications of the Microlensing Observations for Gravitational Theory

With the advent of the microlensing observations of the OGLE (Udalski et al 1993, 1994), MACHO (Alcock et al 1993) and EROS (Aubourg et al 1993) collaborations it became possible to explore not only whether the presumed dark matter spherical halo actually exists, but also to address the critical issue we raised in Sec. (3) regarding what the actual magnitude of the mass to light ratio of a visible disk might be. Neither microlensing off the LMC nor the Milky Way optical searches of the recently refurbished Hubble Space Telescope (Bahcall et al 1994, Paresce, De Marchi and Romaniello 1995) are so far finding the copious amounts of conventional astrophysical dark or faint matter that had been widely anticipated to reside in the halo prior to these observations, while, to the complete contrary, microlensing off the bulge of the Galaxy is finding an unexpectedly large number of such sources in the plane of the Galaxy. The extreme (but not yet unequivocal) interpretation of these data is that there is little or no baryonic halo at all and that the inner region optical disk is maximal with the largest possible  $M/L$  ratio, i.e. that it is precisely of noneother than the very structure required in Sec. (3) of the conformal gravity theory fits. (Given the fact that the data do also permit of some form of halo, albeit at a lower density than that favored by the dark matter models, we note in passing that our current lack of knowledge as to the explicit parameters of any such halo leaves us momentarily in the unsatisfactory position of not being able to do precision fitting to galactic rotation curves, in any theory of rotation curves in fact.)

Apart from not actually finding much if any of a spherical halo at all, the general systematics of what has in fact been found in the plane of the Galaxy now creates several quite severe new challenges for the

standard theory. First, the very presence of all these microlensing sources in the plane of the Galaxy makes a Newtonian disk even less stable than before, thus requiring even more halo matter again to stabilize the Galaxy just at a time when no such halo dark matter is being found. Second, if there is still to be a halo, then it must now be predominantly non-baryonic and that (unlike the situation in typical dark matter fits to dwarf galaxies where the dark matter halo usually does contribute in the inner region - see e.g. Begeman, Broeils and Sanders 1991) the halo must now make no contribution in the inner region since maximal disk luminous matter already exhausts the velocity there. Thus any non-baryonic dark halo must be clever enough to keep out of the optical disk region even as it stabilizes it, and also to normalize itself each and every time to the luminosity in that selfsame optical disk so as to still yield Tully-Fisher, something which would not seem to be immediately apparent for putative weakly interacting non-baryonic wimps. Third, given a potentially maximal Milky Way disk, it now becomes very hard to understand why the  $M/L$  ratios of the luminous disks in dwarf galaxies should be as low as they have in fact been found to be in dark matter fits (reported values are at least an order of magnitude lower than those associated with regular spirals) especially since the stellar populations of dwarfs are not that different from those of regular spirals. Or stated differently, the dwarfs now appear to have a problem not only of too little disk luminous mass (in the outer region) but also one of too much disk luminous mass in the inner region (which is then conveniently finessed by arbitrarily cutting down the associated disk mass to light ratios in the fits, i.e. by effectively treating the matter in the disk as though it also had a repulsive gravitational component), with the luminous dwarf disks simply producing too much gravity for the dark matter fits to handle. Now it is worth noting that the dwarf galaxy fits actually appear to fall into two categories. There are some for which a maximal disk gives acceptable fitting (with a less than maximal one and a consequently bigger inner region halo then just giving better fitting), and there are some for which maximal disks fail completely in both the shape and the normalization of the inner region rotation curves. Thus something has to give somewhere, and it would therefore appear to be worthwhile to again measure the surface brightnesses of dwarf galaxies, optimally in many filters, to see if some optical components have been missed or if perhaps some scale length values might change, so that the inner regions of dwarf galaxies might then in fact be fitted with typical maximal spiral disk  $M/L$  ratios after all, so that they then would in fact be compatible with the disk microlensing data.

As regards the conformal linear potential theory, we already noted that it appears capable of reproducing all the desirable aspects of galactic dark matter without needing the dark matter itself, and now we see that it also appears to have survived the microlensing observations unscathed. Thus it must indeed be regarded as viable. Given the success (so far) of the linear potential theory in fitting rotation curve and cluster data without needing to invoke dark matter, it would thus appear to us that at the present time one cannot categorically assert that the sole gravitational potential on all distance scales is the Newtonian one; and that, in the linear potential, the standard  $1/r$  potential would not only appear to have a companion but to have one which would even dominate over it asymptotically. Indeed, the very need for dark matter in the standard theory may simply be due to trying to apply just the straightforward Newtonian potential in a domain for which there is no prior (or even current for that matter) justification. Even though the observational confirmation on terrestrial to solar system distance scales of both the Newton theory and its general relativistic Einstein corrections technically only establishes the validity of the Newton-Einstein theory on those scales, nonetheless, for most workers in the field, it seems to have established the standard theory on all other distance scales too; despite the fact that many other theories could potentially have the same leading perturbative structure on a given distance scale and yet differ radically elsewhere. Since we have shown that the conformal theory also appears to be able to meet the constraints of data, one has to conclude that at the present time the Newton-Einstein theory is only sufficient to describe data, but not yet necessary. Indeed, it is the very absence of some principle which would single out the Einstein theory from

amongst all other possible covariant theories which one could in principle at least consider which prevents the Einstein theory from yet being a necessary theory of gravity. In fact, in a sense, it is the absence of some underlying principle which would ensure its uniqueness that is the major theoretical problem for the Einstein theory, rather than its phenomenological inability to fit data without invoking dark matter; with this very lack itself actually opening the door to other contenders (Mannheim 1994).

To conclude this paper we would like to state that since the great appeal of Einstein gravity is in its elegance and beauty, using an approach as ad hoc and contrived as dark matter for it almost defeats the whole purpose, and would even appear to be at odds with Einstein's own view of the way nature works. Indeed, Einstein always referred to the Einstein Equations as being a bridge between the beautiful geometry of the Einstein tensor and the ugliness of the energy-momentum tensor. The dark matter idea only serves to make the energy-momentum tensor even more ugly, and could even be construed as a reinvention of the aether. The great aesthetic appeal of the conformal theory is that it adds beauty to both sides of the gravitational equations of motion by both retaining covariance and by endowing both the sides of the bridge with the additional, highly restrictive, symmetry of conformal invariance; and, as we have seen, such a theory may even be able to eliminate the need for dark matter altogether.

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## Figure Captions

Figure (1). The calculated rotational velocity curves associated with the conformal gravity potential  $V(r) = -\beta c^2/r + \gamma c^2 r/2$  for the four representative galaxies, the intermediate sized NGC 3198, the compact bright NGC 2903, the large bright NGC 5907, and the dwarf irregular DDO 154 (at two possible adopted distances). In each graph the bars show the data points with their quoted errors, the full curve shows the overall theoretical velocity prediction (in km/sec) as a function of distance (in arc minutes) from the center of each galaxy, while the two indicated dotted curves show the rotation curves that the separate Newtonian and linear potentials would produce when integrated over the luminous matter distribution of each galaxy. No dark matter is assumed.

Figure (2). The flattest possible rotation curve for a thin exponential disk of stars each with conformal gravity potential  $V(r) = -\beta c^2/r + \gamma c^2 r/2$  which is obtained when the dimensionless ratio  $\eta$  takes the value 0.069. The full curve shows the overall theoretical velocity prediction (in units of  $v/v_0$ ) as a function of distance (in units  $R/R_0$ ), while the two indicated dotted curves show the rotation curves that separate Newtonian and linear potentials would produce. In the upper diagram the rotation curve is plotted out to 10 scale lengths to fully exhibit its flatness, while in the lower diagram it is plotted out to 15 scale lengths to exhibit its eventual asymptotic rise.

Figure (3). The projected line of sight velocity data for the Coma cluster (as binned by The and White) plotted as function of impact parameter distance (in arc min) from the center of the cluster.

Figure (4). The Newtonian potential expectation for  $\langle \sigma_p^2(R) \rangle^{1/2}$  for the generic modified Hubble profile matter distribution of Eq. (78) with cut-off  $R_M = 20R_0$  in the pure isotropic, pure circular and pure radial velocity cases. The velocity is normalized to  $(N\beta_{gal}c^2/R_0)^{1/2}$  and the impact parameter distance is plotted in units of the core radius  $R_0$ .

Figure (5). The linear potential expectation for  $\langle \sigma_p^2(R) \rangle^{1/2}$  for the generic modified Hubble profile matter distribution of Eq. (78) with cut-off  $R_M = 20R_0$  in the pure isotropic, pure circular and pure radial velocity cases. The velocity is normalized to  $(N\gamma_{gal}c^2/R_0)^{1/2}$  and the impact parameter distance is plotted in units of the core radius  $R_0$ .

Figure (6). The fractional amount of matter within a given volume of radius  $r$ , and the fractional amount of matter within a given surface of impact parameter  $R$ , both calculated for the generic modified Hubble profile matter distribution of Eq. (78) with cut-off  $R_M = 20R_0$ . The respective distances ( $r$  and  $R$ ) are both plotted (on the same axis) in units of the core radius  $R_0$ .

Figure (7). The partial root mean square average projected line of sight virial velocity  $\sigma_p(r_m)$  associated with Eq. (81) plotted as a function of radial distance from the center of the cluster in the Newtonian and linear cases for the generic modified Hubble profile matter distribution of Eq. (78) with cut-off  $R_M = 20R_0$ . The Newtonian potential velocity is normalized to  $(N\beta_{gal}c^2/R_0)^{1/2}$ , the linear potential velocity is normalized to  $(N\gamma_{gal}c^2/R_0)^{1/2}$ , and the radial distance is plotted in units of the core radius  $R_0$ .

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